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DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

M.Sc. MATHEMATICS

II YEAR

COMPLEX ANALYSIS

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M.Sc. MATHEMATICS –II YEAR
SMAM31: COMPLEX ANALYSIS
SYLLABUS

UNIT-I:

ANALYTIC FUNCTIONS: Analytic functions – Polynomials – Rational functions – Power Series.

Chapter 1: Section 1: 1.1-1.4

UNIT-II:

CAUCHY'S INTEGRAL FORMULA and LOCAL PROPERTIES OF ANALYTICAL FUNCTIONS: The Index of a point with respect to a closed curve – The Integral formula – Higher derivatives. Removable Singularities-Taylors's Theorem – Zeros and poles – The local Mapping – The Maximum Principle.

Chapter 2: Section 2: 2.1 - 2.7

UNIT-III:

THE GENERAL FORM OF CAUCHY'S THEOREM and THE CALCULUS OF RESIDUES: Chains and cycles- Simple Continuity - Homology - The General statement of Cauchy's Theorem - Proof of Cauchy's theorem - Multiply connected regions – The Residue theorem - The argument principle.

Chapter 3: Section 3: 3.1 - 3.8

UNIT-IV:

EVALUATION OF DEFINITE INTEGRALS AND HARMONIC FUNCTIONS: Evaluation of definite integrals - Definition of Harmonic function and basic properties – The Mean value property - Poisson formula.

Chapter 4: Section 4: 4.1 - 4.4

UNIT-V:

HARMONIC FUNCTIONS AND POWER SERIES EXPANSIONS:

Schwarz Theorem - The reflection principle Weierstrass's Theorem – The Taylor's Series – The Laurent series.

Chapter 5: Section 5: 5.1 – 5.5

Recommended Text:

Lars V. Ahlfors, *Complex Analysis*, (3rd edition) McGraw Hill Co., New York, 1979

SMAM31: COMPLEX ANALYSIS

CONTENTS

UNIT I

ANALYTIC FUNCTIONS: Analytic functions – Polynomials – Rational functions – Power Series.

Chapter 1: Section 1: 1.1-1.4

1.Introduction to the Concept of Analytic Function:

When stepping up to complex numbers we have to consider four different types of functions

i) Real functions of real variable
ii) Real functions of Complex variable
iii) Complex functions of real variable
iv) Complex functions of Complex variable

To indicate a complex function of a complex variable we use the notation w=f(z).

The notation y=f(x) is used in a neutral manner with the understanding that x and y can be either real (or) complex.

Definition: Limit

The function f(x) is said to have limit A as $x \longrightarrow a$.

(i.e.,) $\lim_{x \longrightarrow a} f(x) = A$ iff for every $\varepsilon > 0$

Their exist a number $\delta > 0$ with the property that $|f(x)-a| < \varepsilon$ for all values of x such that $|x-a| < \delta$ and $x \neq a$.

Note:

1. $\lim_{x \longrightarrow a} f(x) = A$ iff $\lim_{x \longrightarrow a} \overline{f(x)} = \bar{A}$
2. $\lim_{x \longrightarrow a} f(x) = A$ iff $\lim_{x \longrightarrow a} Ref(x) = Re\,(A)$ and $\lim_{x \longrightarrow a} Im\, f(x) = Im\,(A)$

Definition: Continuity

The function f(x) is said to be continuous at ‘a’ iff $\lim_{x \to a} f(x) = f(a)$

Note:

1. If f(x) and g(x) are continuous then the sum f(x)+g(x) and the product f(x) g(x) are continuous.
2. Also, the quotient $\frac{f(x)}{g(x)}$ is defines and continuous at a iff g(a) $\neq 0$.
3. If f(x) is continuous, the Re f(x) Im f(x) and |f(x)| are also continuous.

Definition:

We define $f'(a) = \begin{cases} \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \\ \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \end{cases}$

Note:

1.A real function of a complex variable either has the derivative zero (or) the derivative does not exist.

Proof:

Let f(z) be a real function of complex variable whose derivative exists at z=a.

Suppose, h $\longrightarrow$ 0 through real values then $f(a+h) - f(a)$ is real and h is real.

$\therefore$ The quotient $\frac{f(a+h)-f(a)}{h}$ is real .

$\therefore f'(a)$ is real.

Suppose, h $\longrightarrow$ 0 through purely imaginary values then $f(a+h) - f(a)$ is real and h is purely imaginary.

$\therefore$ The quotient $\frac{f(a+h)-f(a)}{h}$ is pure imaginary.

Hence, $f'(a) = 0$ (Since, 0 is the only number which is real and purely imaginary).

∴ The real function of Complex Variable either has the derivative zero (or) the derivatis does not exist.

2.The case of complex function of a real variable is reduced to the real case. If we write $z(t) = x(t) + iy(t)$ and the existence of $z'(t)$ is equivalent to the simultaneous existence of $x'(t)$ and $y'(t)$.

1.1. Analytic Function:

Definition: Analytic function on holomorphic function:

Let $f(z)$ be a complex function of a complex variable which possess a derivative wherever the function is defined. Then $f(z)$ is called an analytic function. (holomorphic function)

Note:

1. The Sum and Product of two analytic functions are again analytic.
2. The quotient $\frac{f(z)}{g(z)}$ of two analytic functions is analytic provided $g(z) \neq 0$.
3. If $f(z)$ is analytic then $f(z)$ is continuous.

Proof:

Consider, $f(x+h) - f(2) = h, \frac{f(z+h)-\mathrm{f(z)}}{h}$

$$\lim_{h\to 0} f(x+h)\, f(z) = \lim_{h\to 0} \mathrm{h}\ \frac{f(z+h) - f(z)}{h}$$
$$= \lim_{h\to 0} h \left[\lim_{h\to x} \frac{f(\mathrm{z}+h) - f(z)}{h}\right]$$
$$= \lim_{h\to 0} h f'(z)$$
$$= 0.f'(z) = 0$$
$$\therefore \lim_{h\to 0} f(z+h) - f(z) = 0$$
$$\therefore \lim_{h\to 0} f(z+h) = f(z)$$

$\therefore f(z)$ is continuous.

4.If $f(z)$ is an analytic function then the real and imaginary parts of an analytic function are harmonic which satisfy the C.R (Cauchy Riemann) equation.

Let $f(z) = u(z) + iv(z)$ be an analytic function $\Rightarrow f'(z)$ exist.

If we choose real values for h, then the imaginary part y is kept constant and the derivative becomes a partial derivative w.r. to x.

$$\therefore f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \ldots\ldots\ldots (1)$$

Similarly, if we choose imaginary values for h, (i.e.,) $h = ik$.
we have,

$$f'(z) = \lim_{k\to 0} \frac{f(z+ik) - f(z)}{ik}$$

$$= \frac{1}{i}\lim_{k\to 0} \frac{f(z+ik) - f(z)}{k}$$

$$= \frac{1}{i}\frac{\partial f}{\partial y} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

$$f'(z) = \frac{1}{i}\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= -\mathrm{i}\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$f'(z) = i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

from (1) & (2)

$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$. which is known as $C \cdot R$ equation in complex form.

Now, equating the real and imaginary part. we get, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

(i.e.,) $U_x = V_y$ and $U_y = -v_x$

which is known as C.R. equations in Cartesian form.

To prove: u and v are harmonic.

We know that, the derivative of an analytic function is itself analytic

By this fact, u and v will have continuous partial derivative of all order, and the mixed derivatives will be equal.

(i.e.,) $\frac{\partial^2 u}{\partial xy} = \frac{\partial^2 x}{\partial y\, \partial x}$

Now, $= \frac{x^2 x}{\partial x^2} + \frac{\partial^2 y}{\partial y^2}$

$$\Delta u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right)$$
$$= \frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial^2 v}{\partial y\, \partial x}$$

$\therefore \Delta u = 0$

$\therefore$ The function u Satisfies the Laplace equation $\Delta u = 0$ is said to be harmonic.

Similarly, $\Delta V = 0$

$\therefore V$ is harmonic

Hence, the real and imaginary parts of an analytic function are harmonic which satisfies the C.R. equation.

5. $|\, f'(z) \int^{2}$ is the Jacobian of u and v with respect to x and y.

Proof: $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \cdot$ (by (1))

$$|f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} . \frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \frac{\partial u}{\partial y}$$

$$|f'(x)|^2 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Hence, $|f'(z)|^2$ is the Jacobian of u and v with respect to x and y.

Definition: Conjugate Harmonic function:

If two harmonic functions u and v satisfy the C.R equations $U_x = V_y$ and $U_y = -V_x$. Then V is the conjugate Harmonic function of u.

Note:

C.R equations can be written as,

$\frac{\partial(-v)}{\partial x} = \frac{\partial u}{\partial y}$ and $\frac{\partial(-v)}{\partial y} = -\frac{\partial u}{\partial x}$. Then we say that u is the conjugate harmonic function of $-v$.

Converse of Note (4)

The function u+iv determined by a pair of conjugate harmonic functions is always analytic. (or) The harmonic function u and v Satisfies $C \cdot R$ equations: Then $u +$ iv is an analytic function.

Proof:

Let $z = x + iy$ and $f(z) = u + iv$.

Given, u and v are harmonic functions of x and y

$\because$ u and v have continuous function first order Partial derivatives.

Applying mean Value theorem on two variable y we get,

$$u(x+h, y+k) - u(x,y) = h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} + \varepsilon_1$$
$$v(x+h, y+k) - v(x,y) = h\frac{\partial v}{\partial x} + k\frac{\partial v}{\partial y} + \varepsilon_2$$

where, the remainders ε_1 and $\varepsilon_2 \to 0$ more rapidly than h+ik in the sense that.

$\frac{\varepsilon_1}{h+ik}$ and $\frac{\varepsilon_2}{h+ik} \to 0$ as $h + ik \to 0$.

consider, $\frac{f(z+h+ik)-f(z)}{h+ik} = \frac{f(x+iy+h+ik)-f(x+iy)}{h+ik}$

$$= \frac{f[x+h+i(y+k)] - f(x+iy)}{h+ik}$$
$$= \frac{u(x+h,y+k) + iv(x+h,y+k) - [u(x,y) + iv(x,y)]}{h+ik}$$
$$= \frac{u(x+h,y+k) - u(x,y) + i[v(x+h,y+k) - v(x,y)]}{h+ik}$$
$$= \frac{h\frac{\partial u}{\partial x} + k\frac{\partial u}{\partial y} + \varepsilon_1 + i\left[h\frac{\partial v}{\partial x} + k\frac{\partial v}{\partial y} + \varepsilon_2\right]}{h+ik}$$
$$= \frac{h\frac{\partial u}{\partial x} + k\left(-\frac{\partial v}{\partial x}\right) + \varepsilon_1 + i\left[h\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x} + \varepsilon_2\right]}{h+ik}$$
$$= \frac{\frac{\partial u}{\partial x}(h+ik) + \frac{\partial v}{\partial x}(-k+ih) + \varepsilon_1 + i\cdot\varepsilon_2}{h+ik}$$
$$= \frac{1}{h+ik}\left[\frac{\partial u}{\partial x}(h+ik) + \frac{\partial v}{\partial x}(i^2k+ih) + \varepsilon_1 + i\varepsilon_2\right]$$
$$= \frac{1}{h+ik}\left[\frac{\partial u}{\partial x}(h+ik) + i\frac{\partial v}{\partial x}(h+ik) + \varepsilon_1 + i\varepsilon_2\right]$$

$$= \frac{(h+ik)}{h+ik}\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varepsilon_1 + i\varepsilon_2}{h+ik}\right]$$
$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varepsilon_1}{h+ik} + i\frac{\varepsilon_x}{h+ik}$$
$$\lim_{h+ik\to 0} \frac{f(z+h+ik) - f(z)}{h+ik} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$= f'(z)$ $\therefore f'(z)$ exists.

$\therefore f(z)$ is an analytic function.

Note:

Conjugate of a harmonic function can be found by integration.

For example:

Let $u = x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial^2 u}{\partial x^2} = 2,\ \frac{\partial^2 u}{\partial y^2} = -2$$
$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$
$$\therefore \Delta u = 0$$

$\therefore u$ is harmonic.

Let v be a conjugate harmonic function of u.

$\therefore u$ and v satisfy $C.R$ equations.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\frac{\partial v}{\partial y} = 2x$ ………. (1) $\frac{\partial v}{\partial y} = 2y$ …………(2)

Integrating equation (1), $\int \partial v = \int 2x \ \partial y$

$\therefore V = 2xy + \phi(x)$

Partially Diff with respect to x,

$$\frac{\partial V}{\partial x} = 2y + \phi'(x)$$
$$\therefore \ 2y = 2y + \phi'(x)$$
$$\therefore \ \phi'(x) = 0$$
$$\phi(x) = \text{ constant } = c$$
$$\therefore V = 2xy + c$$

The analytic function $f(z) = u + iv$

$$= x^2 - y^2 + i[2xy + c]$$
$$= x^2 - y^2 + i2xy + ic$$
$$= (x + iy)^2 + ic$$
$$\therefore f(z) = z^2 + ic$$

Problems:

1.Verify C.R equation for the function z^2 and z^3

Proof:

i) Let $f(z) = z^2$

$= (x + iy)^2$.

$f(\text{z}) = x^2 - y^2 + 2ixy$

$u = x^2 - y^2$; $v = 2xy$

$$\frac{\partial u}{\partial x} = 2x\,, \quad \frac{\partial v}{\partial x} = 2y$$
$$\frac{\partial u}{\partial y} = -2y\,, \qquad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., $U_x = V_y$ and $U_y = -V_x$

ii) Let $f(\text{z}) = z^3$

$= (x + iy)^3$

$= x^3 + (iy)^3 + 3x^3iy + 3x(iy)^2$

$= x^3 - iy^3 + 3ix^3y - 3xy^2$

$f(z) = x^3 - 3xy^2 - i(y^3 - 3x^2y)$

$u = x^3 - 3xy^2$, $v = -y^3 + 3x^2y$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6yx$, $\frac{\partial v}{\partial y} =$ -$3y^2 + 3x^2$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - y^2 = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, $u_x = v_y$ and $u_y = -v_x$

Hence, the C.R equations for the functions z^2 and z^3.

2. Show that an analytic function cannot have a constant absolute value without reducing to a constant.

i.e., An analytic function with constant modulus reduces to a constant.

Solution:

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function with constant modulus.

$$\therefore |f(z)| = \sqrt{u^2 + v^2} = c$$
$$\therefore u^2 + v^2 = c^2$$

partially diff w. r, to x

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0 \quad \ldots\ldots\ldots\ldots (1)$$

partially diff w.r.to y

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0$$

$$\Rightarrow -u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}=0$$

$$\Rightarrow u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eliminate $\frac{\partial u}{\partial x}$ from (1) & (2)

$$(1)\ multiply\ v \Rightarrow \quad uv\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial x}=0$$

$$(2)\ multiply\ u \Rightarrow -uv\frac{\partial u}{\partial x}+u^2\frac{\partial v}{\partial x}=0$$

$$(u^2+v^2)\ \frac{\partial v}{\partial x}=0$$

$$u^2+v^2\ =0\ \text{(or)}\ \frac{\partial v}{\partial x}=0 \quad \ldots\ldots\ldots\ldots (3)$$

Eliminate $\frac{\partial v}{\partial x}$ from (1) & (2)

$$(1)\ multiply\ u \Rightarrow u^2\frac{\partial u}{\partial x}+uv\frac{\partial v}{\partial x}=0$$

$$(2)\ multiplyv \Rightarrow -v^2\frac{\partial u}{\partial x}+uv\frac{\partial v}{\partial x}=0$$

$$(u^2 + v^2)\frac{\partial u}{\partial x} = 0$$

$$u^2 + v^2 = 0 \text{ (or) } \frac{\partial u}{\partial x} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (4)$$

From (3) & (4), $u^2 + v^2 = 0$ (or) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$

Suppose, $u^2 + v^2 = 0$

$\therefore |f(z)| = 0$

which is not possible.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$$

Also, We know that $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$

$\therefore f(z)$ is constant.

Hence, an analytic function with constant modulus reduces to a constant.

3.Prove that the function $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

Proof:

Let $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$ be an analytic function

$\therefore u$ and v have continuous first order partial derivatives which satisfy $C \cdot R$ equation

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

To prove: $\overline{f(\bar{z})}$ is analytic.

Let $\bar{z} = x - iy$

$f(\bar{z}) = u(x,-y) + iv(x,-y)$
$\overline{f(\bar{z})} = u(x,-y) - iv(x,-y)$
$= u_1(x,y) + iv_1(x,y)$

where $u_1(x,y) = \text{u}(x,-y)$

$v_1(x,y) = -v(x,+y)$

$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x}$ & $\frac{\partial u_1}{\partial y} = -\frac{\partial v}{\partial y}$ ………… (1)

$\frac{\partial v_1}{\partial x} = -\frac{\partial v}{\partial x}$ & $\frac{\partial v_1}{\partial y} = \frac{\partial v}{\partial x}$ ………….. (2)

To prove $\frac{\partial u_1}{\partial x} = \frac{\partial v_1}{\partial y}$ $\frac{\partial u_1}{\partial y} = -\frac{\partial v_1}{\partial x}$ …………(3)

$$\therefore \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u_1}{\partial y}$$
$$\frac{\partial u_1}{\partial y} = \frac{-\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial v_1}{\partial x}$$

$\because u_1$ & v_1 Satisfy the C.R. equations.

Also, u_1 and v_1 have continuous first order partial derivatives (Since, u and v have continuous first order partial derivatives).

$\therefore \overline{f(\bar{2})}$ is analytic.

Conversely,

Let $\overline{f(\bar{z})}$ be analytic.

using the first part we get $\overline{\overline{f(\bar{z})}}$ is analytic.

(i.e.,) $f(z)$ is analytic.

Hence, the function $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously analytic.

4.Prove that the function $u(z)$&$U(\bar{z})$ are Simultaneously harmonic.

Proof:

Let $U(z) = U(x, y)$ be a harmonic function.

$\therefore \frac{\partial^2 U(\text{z})}{\partial x^2} + \frac{\partial^2 U(\text{z})}{\partial y^2} = 0$

To prove: $U(\bar{z})$ is harmonic.

(i) To prove: $\frac{\partial^2 U(\bar{z})}{\partial x^2} + \frac{\partial^2 (U\bar{z})}{\partial y^2} = 0.$

Now, $U(\bar{z}) = U(x, -y)$

$\frac{\partial U(\bar{z})}{\partial x} = \frac{\partial u}{\partial x}$ & $\frac{\partial^2 u(\bar{z})}{\partial x^2} = \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u(\bar{z})}{\partial y} = -\frac{\partial u}{\partial y}$ & $\frac{\partial^2 u(\bar{z})}{\partial y^2} = \frac{\partial^2 u}{\partial y^2}$

$\therefore \partial^2 u(\bar{z}) + \frac{\partial^2 u(\bar{z})}{\partial y^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore U(\bar{z})$ is harmonic.

Conversely,

Suppose that, $U(\bar{z})$ is harmonic using the first part, $U(\overline{\bar{z}})$ is harmonic.

$\therefore U(z)$ is harmonic

Hence, the function $U(z)$ and $U(\bar{z})$ are Simultaneously harmonic.

Note:

1.Consider, a complex function $f(x, y)$ of two veal variables. Introduce two complex variable z and $\bar{z}$ as follows.

$x = \frac{1}{2}(z + \bar{z})$ & $y = \frac{1}{2i}(z - \bar{z})$ with this change of variable we can consider, $f(x, y)$ as a function, of z&$\bar{z}$, which we will treat as independent variables. Suppose, f is analytic.

consider, $\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$

$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{-1}{2i}\right)$

$= \frac{1}{2i}\frac{\partial f}{\partial y} - \frac{1}{2i}\frac{\partial f}{\partial y}$ (using C.R equation in complex form)

$\frac{\partial f}{\partial z} = 0$

∴ The analytic functions are characterised by the condition $\frac{\partial f}{\partial \bar{z}} = 0$.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial z}$$
$$= \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{1}{2i}\right)$$
$$= \frac{1}{2i}\frac{\partial f}{\partial y} + \frac{1}{2i}\frac{\partial f}{\partial y} \neq 0$$
$$\therefore \frac{\partial f}{\partial z} \neq 0$$

∴ An analytic Function is independent of $\bar{z}$ and a function of z alone.

2.Without wee of integration the analytic function $f(\mathrm{Z})$ whose real port is given harmonic function $U(x, y)$ can be formed ad follows,

consider, the conjugate function $\overline{f(z)}$

By the above note, $\frac{\overline{\partial f(z)}}{\partial z} = 0$

i.e.,) $\overline{f(z)}$ can be considered as function of $\bar{z}$ alone

$$U(x, y) = \frac{f(z) + \overline{f(z)}}{2}$$
$$= \frac{f(z) + \tilde{f}(\bar{z})}{2} \left(\because \text{ Denote the function } \overline{f(z)} \text{ by } \tilde{f}(\bar{z})\right)$$
$$= \frac{f(x + iy) + \tilde{f}(x - iy)}{2}$$

Put $x = \frac{z}{2}$ & $y = \frac{z}{2i}$

$$u\left(\frac{z}{2},\frac{z}{2\mathrm{i}}\right)=\frac{f\left(\frac{z}{2}+\mathrm{i}\frac{z}{2\mathrm{i}}\right)+\tilde{f}\left(\frac{z}{2}-\mathrm{i}\frac{z}{2\mathrm{i}}\right)}{2}$$

$$=\frac{f(z)+\tilde{f}(0)}{2}$$

$$u\left(\frac{z}{2},\frac{z}{2\mathrm{i}}\right)=\frac{f(z)+\tilde{f}(0)}{2} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

Assume, $\bar{f}(0)$ is real. $(\because f$ is real $f=\tilde{f})$

$$\therefore u(0,0)=\frac{f'(0)+\tilde{f}(0)}{2}=\frac{\tilde{f}(0)+\tilde{f}(0)}{2}$$
$$=\tilde{f}(0)$$

Substitute in equation (1), we get

$$u\left(\frac{z}{2},\frac{z}{2\mathrm{i}}\right)=\frac{f(z)+u(0,0)}{2}$$
$$\therefore f(z)=2u\left(\frac{z}{2},\frac{z}{2\mathrm{i}}\right)-u(0,0)$$

1.2. Polynomials

Every constant is an analytic function with derivative zero. The function z is also an analytic function with derivative one.

Since the sum and product of two analytic functions are analytic, it follows that every Polynomial.

$P(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n$ is an analytic function.

It's derivative is $p'(z)=a_1+2a_2z+\cdots+h_{a_n}z^{n_1}$

By the fundamental theorem of algebra. For $n>0$ the equation $P(z)=0$ has at least one root.

If $P(\alpha_1)=0$, then we can write, $P(z)$ as $P(z)=(z-\alpha_1)P_1(z)$ where $P_1(z)$ is a polynomial of degree $(n-1)$

Repeating this process, $p(z)$ can be written as $P(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$

where $\alpha_1, \alpha_2 \dots \alpha_n$ are not necessarily distinct. These α_j 's are called zeros of the polynomial $P(z)$.

If exactly h of $\alpha_j's$ are coincide then a common value is called the zero of the polynomial $P(z)$ of order h.

Note:

The order of a zero ' α ' can be deformined by consideration of the successive derivatives of p(z) for $z = \alpha$. Suppose, α is a zero of order h.

Than we can write, $P(z) = (z - \alpha)^h P_h(z)$ where $P_h(\alpha) \neq 0$

Successive derivation yields. $P(\infty) = p'(\alpha) =. \cdots = p^{(h-1)}(\alpha) = 0$. while $p^{(h)}(\alpha) \neq 0$.

In other words, the order of a zero equals the order of the First non-vanishing derivative. Definition: Simple zero

A zero of order one is called a simple zero and is characterized by the conditions $p(\alpha) = 0$ and $P'(\alpha) \neq 0$.

Theorem 1: (Lucas Theorem)

If all zeros of a polynomial $p(z)$ lie in a half plane, then all zeros of the derivative $p'(z)$ lie in the same half plane.

Proof:

Let $p(z) = a_n(z - \alpha_1)(z - \alpha_1) \cdots (z - \alpha_n)$

Taking log on both sides,

$\therefore \log p(z) = \log a_n + \log(z - \alpha_1) + \cdots + \log(z - \alpha_n)$

diff with respect to ' z '

$\frac{p'(z)}{p(z)} = \frac{1}{z-\alpha_1} + \frac{1}{z-\alpha_2} + - \cdots + \frac{1}{z-\alpha_n}$ .... (1)

Now, $\alpha_1, \alpha_2 \ldots \alpha_n$ are the zeroes of the polynomial $p(z)$.

Suppose $\alpha_1, \alpha_2 \ldots \alpha_n$ lie in the half plane H. i.e.,) if $\alpha_k \in H, \forall\ k = 1,2, \ldots n$

Then $\text{Im}\left(\frac{\alpha_k - a}{b}\right) < 0$

Let $z \notin H$, then $\text{Im}\left(\frac{z-a}{b}\right) \geq 0$

Now, $\text{Im}\left(\frac{z-\alpha_t}{b}\right) = \text{Im}\left(\frac{z-a+a-\alpha k}{b}\right)$

$$= \text{Im}\left(\frac{z-a}{b} - \frac{\alpha_k - a}{b}\right)$$
$$= \text{Im}\left(\frac{z-a}{b}\right) - \text{Im}\left(\frac{\alpha_k - a}{b}\right)$$
$$> 0$$

$$\therefore \text{Im}\left(\frac{z-\alpha_k}{b}\right) > 0$$

$\therefore \text{Im}\left(\frac{b}{z-\alpha_k}\right) < 0$ .... (2)

From (1) $\frac{bp'(z)}{p(z)} = \frac{b}{z-\alpha_1} + \frac{b}{z-\alpha_2} + \cdots + \frac{b}{z-\alpha_n}$

$$= \sum_{k=1}^{n} \frac{b}{z-\alpha_k}$$

$$\text{Im}\left(\frac{bp'(z)}{p(z)}\right) = \text{Im}\left(\sum_{k=1}^{n} \frac{b}{z-\alpha_k}\right) = \sum_{k=1}^{n}\left(\text{Im}\frac{b}{z-\alpha_k}\right) < 0 \quad (\text{ by } (2)\ )$$

$$\because \text{Im}\left(\frac{bp'(z)}{P(z)}\right) < 0$$
$$\frac{bp'(z)}{p(z)} < 0$$
$$\Rightarrow\ p'(z) \neq 0$$
$$z \notin H\ \Rightarrow\ p'(z) \neq 0$$

Taking negation, $\therefore p'(z) = 0 \Rightarrow z \in H$

1.3. Rational Functions:

Consider, a rational function $R(z) = \frac{P(z)}{Q(z)}$ given as a quotient of two polynomials. We assume $P(z)$ & $Q(z)$ have no common factors and hence no common zeros.

$R(z)$ will be given the value ∞ at zero's of $Q(z)$.

$\therefore R(z)$ is considered as a function in the extended complex plane.

The zeros of $Q(z)$ are called poles of $R(z)$ and the order of the pore of $R(z) =$ the order of corresponding zeros of $Q(z)$

Note:

1.The derivative $R'(z) = \frac{Q(z)P'(z)-P(z)Q'(z)}{Q(z)^2}$ exist only when $Q(z) \neq 0$.

We note that, $R'(z)$ hat the some poles as $R(\mathrm{z})$ and the order of each pole being increased by one.

2.Let $R(z) = \frac{P(z)}{Q(z)} = \frac{a_0+a_1z+a_2z^2+\cdots+a_nz^{\mathrm{n}}}{b_0+b_1z+b_2z^2+\cdots+b_nz^{\mathrm{n}}}$ be the given rational function.

Then there are exactly n-zeros and n-poles in the finite part of the plane.

Problem 1:

Find the order of the zero (or) a pole at ∞

Solution:

Consider, $R(1/\mathrm{z})$

We write as a rational function $R_1(z)$

(i.e.,) $R(1/\mathrm{z}) = R_1(\mathrm{z})$

Set $R(\infty) = R_1(0)$

IF $R_1(0) = 0$ (or) ∞, the order of the zero (or) pole at ∞ is defined as order of the zero (or) pole of $R_1(z)$ at the origin.

[Since, the behaviour of $R(z)$ at ∞ is Same as behaviour of $R_1(z)$ at the origin].

Now, $R_1(z) = R(1/z) = \frac{P(1/z)}{Q(1/z)}$

$$= \frac{a_0 + a_1(1/z) + a_2(1/z^2) + \cdots + a_n(1/z^n)}{b_0 + b_1(1/z) + b_2(1/z^2) + \cdots + b_m(1/z^m)}$$

$$= \frac{(a_0 z^n + a_1 z^{n-1} + \cdots + a_n)/z^n}{(b_0 z^m + b_1 z^{m-1} + \cdots + bm)/z^m}$$

$$= z^{m-n} \frac{(a_0 z^n + a_1 z^{n-1} + \cdots + a_n)}{b_0 z^m + b_1 z^{m-1} + \cdots + b_m}$$

i) If $m > n$, origin becomes a zero of $R_1(z)$ with order $m - n$.

$\therefore \infty$ is a pole of $R(z)$ with order $m - n$.

ii) If $m < n$, origin becomes a pole of $R_1(z)$ with order $n - m$.

$\therefore \infty$ is a zero of $R(z)$ with order $n - m$.

iii) If $m = n, R_1(z) = \frac{a_n}{b_m} \neq 0$ (or) ∞. $\therefore \infty$ is neither zero nor pole of $R(z)$

	No. of zeros in the finite part of the plane	No. of zeros at ∞	Total	No. of poles in the finite point of the plane	No. of poles at ∞	Total
$m > n$	n	$m - n$	m	m	-	m
$m < n$	n	-	n	m	$n - m$	n
$m = n$	n	-	n	m	-	m

The above table shows that, number of zeros and number of poles in the extended complex plane are the same and it is equal to greater m and n.

This common number of ceros and poles is called the order of the rational function $R(z)$.

Note:

1. A rational function $R(z)$ with order p has p zeros and P poles and every equation $R(z) = a$, where ' a ' is a constant has exactly p ' roots.

2. Every rational function has a representation by Partial fractions.

Assume that $R(z)$ has a pole at infinity we Carryout the division of $\frac{P(z)}{Q(z)}$ until the degree of remainder is almost equal to that of the: denominator.

The result can be written in the form $R(z) = G(z) + H(z)$ (1), where $G(z)$ is a polynomial without constant term and $H(z)$ is finite at infinity. The degree of $G(z)$ is the order of the pole at infinity and the polynomial $G(z)$ is called the Singular part of $R(z)$ at infinity. Let the distinct Finite poles of $R(z)$ be denoted by $\beta_1, \beta_2 \cdots \beta_q$.

The function $R\left(\beta_j + \frac{1}{\varepsilon}\right)$ is a rational function of ε with a pole at $\varepsilon = \infty$.

By use of the decomposition (1) we can write $R\left(B_j + \frac{1}{\varepsilon}\right) = G_j(\varepsilon) + H_j(\varepsilon)$ (on with a change of Variably

$$R(z) = G_j\left(\frac{1}{z - \beta_j}\right) + H_j\left(\frac{1}{z - \beta_j}\right)$$

Here, $G_j\left(\frac{1}{2-\beta_j}\right)$ is a polynomial in $\frac{1}{2-\beta_j}$ with out constant term, called the Singular port of $R(z)$ at β_j.

The function $H_j\left(\frac{1}{z-\beta_j}\right)$ is finite for $z = \beta_j$.

Consider, the expression $R(z) - G(z) - \sum_{j=1}^{2} G_j\left(\frac{1}{z-R_j}\right)$.

This is a rational function which cannot have poles other than $\beta_1, \beta_2 \ldots \beta_q \& \infty$.

At $z = B_j$ we find that the two terms which becomes ∞ have a difference $H_j\left(\frac{1}{2-\beta_j}\right)$ with a

forite

limit and the same is true at ∞.

$\therefore$ (2) has neither any finite poles nor a pole at in fixity.

$\therefore$ The above rational Function reduces to a constant. If this constant is absorbed in $G(z)$.

we obtain, $R(z) = G(z) + \sum_{j=1}^{q} G_j\left(\frac{1}{2-\beta_j}\right)$.

Problem 2:

Use the method of the text to developer

i) $\frac{z^4}{z^3-1}$ in partial Fraction.

Solution:

i) Let $R(z) = \frac{z^4}{z^3-1}$

The poles are got by, $z^3 - 1 = 0$

$\Rightarrow z^3 = 1$
$\Rightarrow z = 1, w, w^2$
$\therefore \beta_1 = 1, \beta_2 = \omega, \beta_3 = w^2$

$\therefore G(z) = z$

Consider, $R(\beta_1 + 1/\varepsilon) = R(1 + 1/\varepsilon)$

$$= \frac{(1 + 1/\varepsilon)^4}{(1 + 1/\varepsilon)^3 - 1}$$

$$= \frac{1 + 4(1/\varepsilon) + 6(1/\varepsilon^1) + 4(1/\varepsilon^3) + 1/\varepsilon^4}{1 + 3(1/\varepsilon) + 3(1/\varepsilon^2) + 1/\varepsilon^3 - 1}$$

$$= \frac{(\varepsilon^4 + 1 - \varepsilon^3 + 6\varepsilon^2 + 4\varepsilon + 1)/\varepsilon^4}{3\varepsilon^2 + 3\varepsilon + 1/\varepsilon^3}$$

$$= \frac{\varepsilon^4 + 4\varepsilon^3 + 6\varepsilon^2 + 4\varepsilon + 1}{\varepsilon(3\varepsilon^2 + 3\varepsilon + 1)}$$

$$\therefore R(\beta_1 + 1/\varepsilon) = \frac{\varepsilon^4 + 4\varepsilon^3 + 6\varepsilon^2 + 4\varepsilon + 1}{3\varepsilon^3 + 3\varepsilon^2 + \varepsilon}$$

$$\therefore G_1(1 + 1/\varepsilon) = 1/\varepsilon^2$$

$$\therefore G_1(z) = \frac{1}{3(z-1)}$$

$$= \frac{(w + 1/\varepsilon)^4}{(w + 1/\varepsilon)^3 - 1}$$

$$= \frac{w^4 + 4w^3(1/\varepsilon) + 6w^2(1/\varepsilon^2) + 4(1/\varepsilon^3)^w + (1/\varepsilon^4)}{w^3 + 3w^2(1/\varepsilon) + 3w(1/\varepsilon^2) + 1/\varepsilon^3 - 1}$$

$$= \frac{\omega + 4(1/\varepsilon) + 6\omega^2(1/\varepsilon^2) + 4w(1/\varepsilon^3) + 1/\varepsilon^4}{1 + 3\omega^2(1/\varepsilon) + 3\omega(1/\varepsilon^2) + 1/\varepsilon^3 - \gamma}$$

$$= \frac{\omega\varepsilon^4 + 4\varepsilon^3 + 6\omega^2\varepsilon^2 + 4w\varepsilon + 1/\varepsilon^4}{3w^2\varepsilon^2 + 3\omega\varepsilon + 1/\varepsilon^3}$$

$$k(\beta_2 + 1/\varepsilon) = \frac{\omega\varepsilon^4 + 1\varepsilon^3 + 6\omega^2\varepsilon^2 + 4\omega\varepsilon + 1}{3\omega^2\varepsilon^3 + 3\omega\varepsilon^2 + \varepsilon}$$

$$\therefore G_2(\omega + 1/\varepsilon) = \frac{1 \cdot \varepsilon}{3\omega}$$

$$z = w + 1/\varepsilon$$

$$= \frac{1}{3\omega(z - \omega)}$$

$$z - \omega = 1/\varepsilon.$$

$$\varepsilon = \frac{1}{z - w}.$$

consider, $R(\beta_3 + 1/\varepsilon) = R(\omega^2 + 1/\varepsilon)$

$$= \frac{(\omega^2 + 1/\varepsilon)^4}{(\omega^2 + 1/\varepsilon)^3 - 1} \quad \left(\because R(z) = \frac{z^4}{z^3 - 1}\right)$$

$$= \frac{(\omega^2)^4 + 4\left(\frac{1}{\varepsilon}\right)(\omega^2)^3 + 6\left(\frac{1}{\varepsilon^2}\right)(\omega^2)^2 + 4(\frac{1}{\varepsilon^3})\omega^2 + 1/\varepsilon^4}{(\omega^2)^3 + 3(1/\varepsilon)(\omega^2)^2 + 3(1/\varepsilon^2)(\omega^2) + 1/\varepsilon^3 - 1}$$

$$= \frac{w^2 + 4\left(\frac{1}{\varepsilon}\right) + 6(1/\varepsilon^2)w + 4\,(\frac{1}{\varepsilon^3}) \cdot w^2 + 1/\varepsilon^4}{1 + 3(1/\varepsilon)w + 3\left(\frac{1}{\varepsilon^2}\right)w^2 + (\frac{1}{\varepsilon^3}) - 1}$$

$$= \frac{\omega^2 \cdot \varepsilon^4 + 4\varepsilon^3 + b\omega\varepsilon^2 + 4\varepsilon\omega^2 + 1/\varepsilon 4}{3\omega\varepsilon^2 + 3\omega^2 \cdot \varepsilon + 1/\varepsilon^3}$$

$$R(\beta_3 + 1/\varepsilon) = \frac{\omega^2\varepsilon^4 + 4\varepsilon^3 + 6\omega\varepsilon^2 + 4\varepsilon\omega^2 + 1}{3\omega\varepsilon^3 + 3\omega^2\varepsilon^2 + \varepsilon}$$

$$\therefore \mathrm{G}_3\left(\omega^2 + \frac{1}{\varepsilon}\right) = \frac{\omega}{3}$$

$[\because z = \omega^2 + 1/\varepsilon$

$\frac{1}{\varepsilon} = z - w^2\,, \quad \varepsilon = \frac{1}{z - w^2}]$

$$\therefore G_3(z) = \frac{\omega}{3(z - \omega^2)}$$

$$\therefore R(z) = G(z) + G_1(z) + G_2(z) + G_3(z)$$

$$R(z) = z + \frac{1}{3(z-1)} + \frac{1}{3\omega(z-\omega)} + \frac{w}{3(z-\omega^2)}$$

ii) Let $R(z) = \frac{1}{z(z+1)^2(z+2)^3}$

The poles are got by, $z(z+1)^2(z+2)^3 = 0$.

$\Rightarrow z = 0 \;\; \& \;\; (z+1)^2 = 0 \;\; \& \; (z+2)^3 = 0$

$\Rightarrow z = 0 \;\&\; z = -1, -1, \;\&\; z = -2, -2, -2$

$\therefore \beta_1 = 0, \beta_2 = -1, \beta_3 = -1, \beta_4 = -2, \beta_5 = -2\,, \beta_6 = -2$

$G(z) = \frac{1}{z(z+1)^2(z+2)^3}$,Consider, $R(\beta_1 + 1/\varepsilon) = R(0 + 1/\varepsilon)$

$$= \frac{1}{(1/\varepsilon)((\frac{1}{\varepsilon}) + 1)^3 \left((\frac{1}{\varepsilon}) + 2\right)^3}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon}\right)\left(\frac{1}{\varepsilon^2} + \frac{2}{\varepsilon} + 1\right)\left(\frac{1}{\varepsilon^3} + 3.2\left(\frac{1}{\varepsilon^3}\right) + 3.4\left(\frac{1}{\varepsilon}\right) + 8\right)}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon}\right)\left(1 + \frac{2}{\varepsilon} + \frac{1}{\varepsilon^2}\right) \cdot \left(\frac{1}{\varepsilon^2} + \frac{6}{\varepsilon^2} + \frac{12}{\varepsilon} + 8\right)}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon} + \frac{2}{\varepsilon^2}\right) + \frac{1}{\varepsilon^3}\Big)\left(\frac{1}{\varepsilon^3} + \frac{6}{\varepsilon^2} + \frac{12}{\varepsilon} + 8\right)}$$

$$= \frac{1}{(\frac{1}{\varepsilon^4} + \frac{6}{\varepsilon^3} + \frac{12}{\varepsilon^2} + \frac{8}{\varepsilon} + \frac{2}{\varepsilon^5} + \frac{12}{\varepsilon^4} + \frac{24}{\varepsilon^3} + \frac{16}{\varepsilon^2} + \frac{1}{\varepsilon^6} + \frac{6}{\varepsilon^5} + \frac{12}{\varepsilon^4} + \frac{8}{\varepsilon^3})}$$

$$R(\beta_1 + 1/\varepsilon) = \frac{\varepsilon^6}{1 + 8\varepsilon + 25\varepsilon^2 + 38\varepsilon^3 + 28\varepsilon^4 + 8\varepsilon^5}$$

$$\therefore G_1(1/\varepsilon) = \frac{1}{8}\varepsilon.$$

$$z = 1/\varepsilon$$

$$\therefore G_1(z) = \frac{1}{8z} \Rightarrow \varepsilon = 1/z.$$

Consider, $R(\beta_1 + 1/\varepsilon) = R\left(-1 + \frac{1}{\varepsilon}\right)$

$$= \frac{1}{\left(-1 + \frac{1}{\varepsilon}\right)\left(-1 + \frac{1}{\varepsilon} + 1\right)^2 \left(-1 + \frac{1}{\varepsilon} + 1\right)^3}$$

$$= \frac{1}{(\frac{1}{\varepsilon} - 1\Big)\frac{1}{\varepsilon^2}\left(\frac{1}{\varepsilon} + 1\right)^3}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon^3} - \frac{1}{\varepsilon^2}\right)\left(\frac{1}{\varepsilon^3} + 3\cdot\frac{1}{\varepsilon^2} + 3\cdot\frac{1}{\varepsilon} + 1\right)}$$

$$= \frac{1}{\frac{1}{\varepsilon^6} + \frac{3}{\varepsilon^5} + \frac{3}{\varepsilon^4} + \frac{1}{\varepsilon^3} - \frac{1}{\varepsilon^5} - \frac{3}{\varepsilon^4} - \frac{3}{\varepsilon^3} - \frac{1}{\varepsilon^2}}$$

$$= \frac{\varepsilon^6}{1 + 3\varepsilon + 3\varepsilon^2 + \varepsilon^3 - \varepsilon - 3\varepsilon^2 - 3\varepsilon^3 - \varepsilon^4}$$

$$\therefore R(\beta_2 + 1/\varepsilon) = \frac{\varepsilon^6}{1 + 2\varepsilon - 2\varepsilon^3 - \varepsilon^4}$$

$$\therefore G_2(- + 1/\varepsilon) = 2\varepsilon - \varepsilon^2$$

$$\therefore G_2(z) = \frac{2}{z+1} - \left(\frac{1}{z+1}\right)^2$$

Consider, $R(\beta_4 + 1/\varepsilon) = R(-2 + 1/\varepsilon)$

$$= \frac{1}{(-2 + 1/\varepsilon)(-2 + (\frac{1}{\varepsilon}) + 1)^2(-2 + (\frac{1}{\varepsilon}) + 2)^3}$$

$$= \frac{1}{\frac{1}{\varepsilon^3}\left(\frac{1}{\varepsilon} - 2\right)\left(\frac{1}{\varepsilon} - 1\right)^2}$$

$$= \frac{1}{\frac{1}{\varepsilon^3}\left(\frac{1}{\varepsilon} - 2\right)\left(\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon} + 1\right)}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon^4} - \frac{2}{\varepsilon^3}\right)\left(\frac{1}{\varepsilon^2} - \frac{2}{\varepsilon} + 1\right)}$$

$$= \frac{1}{\left(\frac{1}{\varepsilon^6} - \frac{2}{\varepsilon^5} + \frac{1}{\varepsilon^4} - \frac{2}{\varepsilon^5} + \frac{4}{\varepsilon^4} - \frac{2}{\varepsilon^3}\right)}$$

$$= \frac{\varepsilon^6}{1 - 2\varepsilon + \varepsilon^2 - 2\varepsilon + 4\varepsilon^2 - 2\varepsilon^3}$$

$$\therefore R(\beta_4 + 1/\varepsilon) = \frac{\varepsilon^6}{1 - 4\varepsilon + 5\varepsilon^2 - 2\varepsilon^3}$$

$$G_4(-2 + 1/\varepsilon) = -\frac{\varepsilon^3}{2} - \frac{5\varepsilon^2}{4} - \frac{17\varepsilon}{8}$$

$$G_4(z) = \frac{-1}{2(z+2)^3} - \frac{5}{1(z+2)^2} - \frac{17}{18(z+2)}$$

$$\therefore R(z) = G_1(z) + G_1(z) + 2G_2(z) + 3G_4(z)$$

$$= \frac{1}{z(z+1)^2(z+2)^3} + \frac{1}{8z} + 2\left(\frac{2}{z+1} - \frac{1}{(z+1)^2}\right) - 3\left(\frac{1}{2(z+2)^2} + \frac{5}{4(z+1)^2} + \frac{1}{8(z+2)}\right)$$

$$\therefore R(z) = \frac{1}{z(z+1)^2(z+2)^3} + \frac{1}{8z} + \frac{4}{z+1} - \frac{2}{(z+1)^2} - \frac{3}{2(z+2)^2} - \frac{15}{4(z+2)^2} - \frac{51}{8(z+2)}$$

1.4. Power series:

Definition: (limit superior)

If A is finite where $A = \lim\sup\{\alpha_n\}$ then given $\varepsilon > 0$ their exist n_0

Such that $|An - A| < \varepsilon, \forall n \geqslant n_0$.

Definition: (absolutely convergent)

A series Σa_n with the property that $\sum|a_n|$ convergesis called absolutely convergent series.

(i.e.) $\sum a_n$ converges and $\sum a_n$ converges then $\sum a_n$ is absolutely convergent

Converges uniformly:

The sequence $\{f_n(x)\}$ converges uniform on $E \Leftrightarrow$ to every $\varepsilon > 0$ their exist an n_0

Such that $|f_m(x) - f_n(x)| < \varepsilon, \forall m, n \geq n_0$ and $\forall x \in E$

Result:

1.Consider the two series $\sum f_n(x)$ and $\sum a_n$ such that $|f_n(x)| \leq Ma_n$, for some constant M for all sufficiently large n.

The first series $\sum f_n(x)$ is called minorant and the 2nd series is called majorant.

2. Weierstrass's - M test:

If $\left|f_n(x) + f_{n+1}(x) + \cdots + f_{m+p}(x)\right| \leqslant M\left(a_n + a_{n+1} + \cdots + a_{n+p}\right)$

and if the majorant (2^{nd} $series$)) converges then the minorant (1st series) converges uniformly.

3. Differentiable at z_0 : (Derivative at z_0)

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

4. $\lim_{n\to\infty} f_n(z) = f(z) \Rightarrow$ for a given $\varepsilon > 0$, their exist an n such that

$|f_n(z) - f(z)| < \varepsilon \quad \forall n.$

Power series:

- The general for of a power series is $\sum_{n=0}^{\infty} a_n z^n$ where ' a_n ' and ' z^n ' are complex.
- $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ represents the power series with respect to the center ' z_0 '.
- consider the geometric series, $1 + z + z^2 + z^3 + \cdots + z^n + \cdots$, the portialsum can be written in the form $1 + z + z^2 + \cdots + z^{n-1} = \frac{1-z^n}{1-z}$

a) If $|z| < 1, |z|^n \to 0$ as $n \to \infty$ and so $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

(i.e.) the geometric series converges if $|z| < 1$

If $|z| \geq 1, |z|^n \to \infty$ as $n \to \infty$ and so $\sum_{n=0}^{\infty} z^n$ diverges to ∞.

Hadamard's formula:

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \text{ (or) } \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$R \to$ Radius of convergence (ROC)

Result:

- The power series $\sum_{n=0}^{\infty} a_n z^n$ convergent if $|z| < R$. (Interior of a circle) and divergent if $|z| > R$. (Exterior of a circle)
- $|z| < 1 \Rightarrow$ $bounded$ $|z| > 1$, unbounded.

Theorem:1 (Abel's theorem)

For every power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a number $R, 0 \leq R \leq \infty$ called the Radius of Convergence, with the following properties:

(i) The series converges absolutely for every z with $|z| < R$. If $0 \leqslant \rho < R$, the convergence is uniform for $|z| \leqslant \rho$

(i) If $|z| > R$, the terms of the series are unbounded and the series is consequently divergent.

(iii) In $|z| < R$, the sum of the series is an analytic function. The derivative can be obtained by term wise, differentiation and the derived series has the same radius of convergence:

Proof:

We know that Hadamards formula is $\frac{1}{R} = \lim\limits_{n\to\infty} \sup \sqrt[n]{|a_n|}$ ....(1)

$R \rightarrow$ Radius of Convergence

Claim: R has the required properties

To prove. (i):

claim:1 $\sum a_n z^n$ converges absolutely if $|z| < R$

(i.e.) To P: $\sum |a_n z^n|$ converges.

given: $|z| < R$

choose e r: $|z| < \rho < R$.

$$\Rightarrow |z| < \rho \text{ and } \rho < R.$$
$$\Rightarrow \frac{|z|}{\rho} < 1 \text{ and } \frac{1}{\rho} > \frac{1}{R}$$

Now, $\frac{1}{\rho} > \frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ (by (1)

By the definition of limit superior, $\exists$ an n_0 such that

$$\Rightarrow \sqrt[n]{|a_n|} < \frac{1}{\rho}$$

(i.e.) $a_n|^{1/n} < 1/e$

$$\Rightarrow |a_n| < \left(\frac{1}{\rho}\right)^n$$

$$\Rightarrow |a_n z^n| < \left(\frac{|z|}{\rho}\right)^n \quad \text{(Multiplyby } |z|^n \text{)}$$

Now $\left(\frac{|z|}{\rho}\right)^n$ is convergent.

(i.e.) The geometric series majorant is convergent.

$$|a_n z^n| < 1$$
$$\Rightarrow \sum |a_n z^n| \text{ converges}$$
$$\Rightarrow \sum a_n z^n \text{ converges absolutely.}$$

claim:2

If $0 \leq \rho \leq R$ & $|z| < R$, then $\sum a_n z^n$ converges uniformly in $|z| \leqslant \rho$.

$0 \leqslant \rho \leqslant R$ and $|z| < R$, also $|z| \leqslant \rho$

choose $\rho' \Rightarrow: 0 \leqslant \rho < \rho' < R$.

$$\Rightarrow \rho < \rho' \text{ and } \rho' < R$$
$$\Rightarrow \frac{\rho}{\rho'} < 1 \text{ and } \frac{1}{\rho'} > \frac{1}{R}$$

$$\frac{1}{\rho'} > \frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} (\text{ by eqn } (1)$$

By definition of limit superior,

their exist an $n_0 = \sqrt[n]{|a_n|} < \frac{1}{\rho'}$

$$\Rightarrow |a_n|^{\frac{1}{n}} < \frac{1}{\rho'}$$
$$\Rightarrow |a_n| < \left(\frac{1}{\rho'}\right)^n$$
$$\Rightarrow |a_n z^n| < \left(\frac{|z|}{\rho'}\right)^n < \left(\frac{\rho}{p'}\right)^n (\because |z| \leq \rho)$$

Now, $\left(\frac{\rho}{\rho'}\right)^n$ is convergent. The geometric series majorant is convergent

$|a_n z^n| < 1$

∴ The majorant converges

Hence By Weierstrass's - M test, $\sum a_n z^n$ converges uniformly.

To prove (ii):

claim:3

If $|z| > R$, the terms of $\sum a_n z^n$ are unbounded and so diverges.

given: $|z| > R$

choose $\rho \ni |z| > \rho > R$

$$\Rightarrow |z| > \rho \text{ and } \rho > R$$
$$\Rightarrow \frac{|z|}{\rho} > 1 \text{ and } \frac{1}{\rho} < \frac{1}{R}$$

$$\frac{1}{\rho} < \frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} \text{ (by (1))}$$

By definition of limit superior, *their exist* an $n_0 \ni$

$$\sqrt[n]{|a_n|} > 1/\rho$$
$$\Rightarrow |a_n|^{\frac{1}{n}} > 1/\rho \Rightarrow |a_n| > (1/\rho)^n$$

$$\Rightarrow |a_n z^n| > \left(\frac{|z|}{\rho}\right)^n > 1$$

The series $\left(\frac{|z|}{\rho}\right)^n$ is unbounded.

$\Rightarrow \sum a_n z^n$ diverges to ∞.

To prove (iii):

Claim:4

We know that, $\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$

The series $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence.

Let R' be the Radius of convergence of $\sum_{n=1}^{\infty} (n a_n) z^{n-1}$

(i.e.) $\frac{1}{R'} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$

(i.e.) $\frac{1}{R'} = \lim_{n\to\infty} \sup n^{1/n} |a_n|^{\frac{1}{n}}$ ……………(I)

To prove: $R = R'$

Now, To prove: $\lim_{h\to\infty} n^{1/n} = 1$

Let $n^{1/n} = 1 + \delta_n$ where $\sigma_n > 0$

$\Rightarrow n = (1 + \delta_n)^n$

By the Binomial theorem,

$$\Rightarrow n = 1 + \binom{n}{1}\delta_n + \binom{n}{2}\delta_n^2 + \cdots + \delta_n^n$$
$$> 1 + \binom{n}{2}\delta_n^2 \text{ (eliminating some term)}$$
$$\Rightarrow n - 1 > \binom{n}{2}\delta_n^2 \ \Rightarrow n - 1 > \frac{n(n-1)}{2}\delta_n^2$$

$$\Rightarrow \frac{2}{n} > \delta_n^2$$
$$\Rightarrow \delta_n < \sqrt{\frac{2}{n}}$$
$$\text{as } n \to \infty, \sqrt{\frac{2}{n}} \to 0$$
$$\Rightarrow \delta_n \to 0 \text{ as } n \to \infty$$
$$\therefore n^{\frac{1}{n}} = 1 + \delta_n = 1 + 0 = 1 \text{ as } n \to \infty$$
$$\Rightarrow n^{1/n} = 1 \text{ as } n \to \infty$$
$$\therefore \lim_{n\to\infty} n^{1/n} = 1$$
$$\therefore \text{ (I) } \Rightarrow \frac{1}{R'} = \lim_{n\to\infty} \sup|a_n|^{1/n}$$
$$= \lim_{n\to\infty} \sup \sqrt[n]{a_n}$$
$$= \frac{1}{R}$$

$\Rightarrow R = R'$, convergent in $|z| \angle R$.

$\Rightarrow \sum_{n=1}^{\infty} a_n z^{n-1}$ is convergent in $|Z| \angle R$.

Claim: 5

If $| Z1 \angle R$, the sum of the series is an analytic function and the derivative is obtained by term wise differentiation.

$$\text{Let } f(z) = \sum_{n=0}^{\infty} a_n z^n = S_n(z) + R_n(z) \quad \ldots\ldots\ldots\ldots(2)$$

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}; \text{ where}$$

$$S_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} \ldots\ldots\ldots\ldots(3)$$

$$= \sum_{k=0}^{n-1} a_k z^k$$

and $R_n(z) = \sum_{n=1}^{\infty} a_k z^k$ .... (4)

Consider the series $\sum_{n=1}^{\infty} n a_n z^{n-1}$.

By claim: 4, this series is convergent in $|z| < \text{R}$.

Let $f_1(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$

$$\Longrightarrow f_1(z) = \lim_{n\to\infty} S_n'(z) \to \text{(II)}$$

At $z = Z_0, f$ an $\varepsilon/3 > 0$

$$|S_n'(z_0) - f_1(z_0)| < \varepsilon/3, \forall n$$

$$\Rightarrow S_n(z_0) - f_1(z_0) < \frac{\varepsilon}{3}, \forall n \quad \ldots\ldots\ldots\ldots(5)$$

$(\because |z| < a$

$\Rightarrow -a < z < a)$

consider $\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)$, where

$|z| < \rho < R$ and $|z_0| < \rho < R$. choose ρ.

$$= \left\{\frac{S_n(z) + R_n(z) - S_n(z_0) - R_n(z_0)}{z - z_0}\right\} - f_1(z_0) \text{ (by (2)}$$

$$= \left\{\frac{S_n(z) - S_n(z_0)}{z - z_0}\right\} + S_n'(z_0) - S_n'(z_0) + \left\{\frac{R_n(z) - R_n(z_0)}{z - z_0}\right\} - f_1(z_0)$$

$$= \left\{\frac{S_n(z)-S_n(z_0)}{z-Z_0}\right\} + [S_n'(z_0) - f_1(z_0)] + \left[\frac{R_n(z)-R_n(z_0)}{z-z_0}\right] \quad \ldots\ldots\ldots\ldots\ldots (6)$$

Now, $\dfrac{R_n(z) - R_n(z_0)}{z - z_0} = \dfrac{\sum_{k=n}^{\infty} a_k z^k - \sum_{k=n}^{\infty} a_k z_0^k}{z - z_0}$

$$= \frac{\sum_{k=n}^{\infty} a_k\left(z^k - z_0^k\right)}{z - z_0}$$

$$= \sum_{k=n}^{\infty} a_n\left(z^{k-1} + z_0 z^{k-2} + z_0^2 z^{k-3} - z_0^{k-1}\right)$$

$$\left|\frac{R_n(z) - R_n(z_0)}{z - z_0}\right| \leq \sum_{k=n}^{\infty} |a_k|$$

given $|z| < \rho, |z_0| < \rho$

$$\leq \sum_{k=n}^{\infty} |a_k| \left[e^{k-1} + \rho'\rho^{k-2} + \rho^2\rho^{k-3} + \cdots + \rho^{k-1}\right]$$

$$\leq \sum_{k=n}^{\infty} |a_k| \left[\rho^{k-1} + \rho^{k-1} + \rho^{k-1} + \cdots + \rho^{k-1}\right](k\text{-times})$$

$$\leqslant \sum_{k=n}^{\infty} |a_k| k\rho^{k-1}$$

∴ The series $\sum_{k=n}^{\infty} k a_k \rho^{k-1}$ is absolutely Convergent at $z = \rho$

∴ their exist an $n_0 \Rightarrow \dfrac{R_n(z) - R_n(z_0)}{z - z_0} < \varepsilon/3 \quad \ldots\ldots\ldots\ldots(7)$

$\forall n \geq n_0$.

By the definition of derivative,

$$\lim_{z \to z_0} \frac{S_n(z) - S_n(z_0)}{z - z_0} = S_n'(z_0)$$

$$\Rightarrow \left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right| < \varepsilon/3 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

$$(6) \Rightarrow \left|\frac{f(z)-f(z_0)}{z-z_0} - f_1(z_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

$$\Rightarrow \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f_1(z_0)$$

$\Rightarrow f'(z_0) = f_1(z_0)$ (by the definition of derivative) at $z = z_0$

$\therefore f$ is differentiable at $z = z_0$.

since z_0 is arbitrary,

$\Rightarrow f$ is analytic.

Problems:

1.Find the radius of convergence for the following power series ($\Sigma a_n z^n$)

(i) $\sum n!\, z^n$ (ii) $\sum \frac{z^n}{n!}$ (iii) $\sum n^p z^n$ (iv) $\sum q^{n^2} z^n$ (v) $z^{n!}$ ($|q|<1$)

Solution:

(i) $\sum n!\, z^n$

Here $a_n = n!$

We know that, $\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{n!}\right|$

$$= \lim_{n\to\infty}\left|\frac{(n+1)n!}{n!}\right| = \lim_{n\to\infty}(n+1) = \infty + 1 = \infty$$

$1/\mathrm{R} = 0 \Rightarrow R = \infty$

(ii) $\sum \frac{z^n}{n!}$

Here $a_n = \frac{1}{n!}$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{(n+1)}\right.}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n!}{(n+1)n!}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)}\right| = \frac{1}{\infty} = 0$$

$\therefore 1/R = 0 \Rightarrow R = \infty$

(iii) $\sum n^p z^n$

Here $a_n = n^p$

$$R = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^p}{n^p}\right|$$
$$= \lim_{n\to\infty}\left|\frac{\left[n\left(1+\frac{1}{n}\right)\right]^p}{np}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n^p\left(1+\frac{1}{n}\right)^p}{n^p}\right|$$
$$= (1+1/\infty)^p = (1+0)^p = 1$$
$$1/R = 1 \Rightarrow R = 1$$

(iv) $\sum q^{n^2} z^n$ Here $a_n = q^{n^2}$

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{q^{(n+1)^2}}{q^{n^2}}\right|$$
$$= \lim_{n\to\infty}\left(\frac{q^{n^2+2n+1}}{a^{n_2}}\right) = \lim_{n\to\infty}\left(\frac{q^{2^2}\cdot a^{2n+1}}{q^{n^2}}\right)$$
$$= \lim_{n\to\infty}(q^{2n+1}) \text{ converges to } 0$$

if $|q| < 1 \therefore \frac{1}{R} = 0 \Rightarrow R = \infty.$

(v) $z^{n!}$

We know that,

$$\sum_{n=0}^{\infty} z^{n!} = z^{0!} + z^{1!} + z^{2!} + \cdots\cdot$$
$$\sum_{n=0}^{\infty} a_n z^n = n = 0 \;= z + z + z^2 + z^6 +$$
$$\sum_{n=0}^{\infty} a_n z^n = 2z + z^2 + z^6 + \cdots\cdots$$
$$\Rightarrow a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots = 2z + z^2 + z^6 + \cdots$$
$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = a_4 = a_5 = 0, a_6 = 1$$

$\therefore a_k = \begin{cases} 1 \text{ if } k = n! \\ \text{elsewhere} \end{cases} \rightarrow (1)\ |a_k| = 1$

$$\frac{1}{R} = \lim_{n\to\infty} \sup|a_n|^{1/n} = 1$$

2. $\sum a_n z^n$ has radius of convergence R. What is the radius of Convergence of $\sum a_n z^{2n}$ and

$\sum {a_n}^2 z^n$.

Solution:

Given $\sum a_n z^n$ has radius of convergence R.

1. $\sum a_n z^{2n} = \sum a_n \omega^n$ where $w = z^2$

$$\sum a_n \omega^n_{=} = \begin{cases} \text{converges} & \text{if } |\omega| < R \\ \text{diverges} & \text{if } |\omega| > R \end{cases}$$

$|\omega| < R \Leftrightarrow |z^2| < R$

((e) converges if $|z^2| \angle R$

diverges if $|z^2| > R$

$\sum a_n z^{2n}$ converges if $|z| < \sqrt{R}$

diverges if $|z| > \sqrt{R}$

$\because$ The Radius of convergence $R = \sqrt{R}$

2. $\sum a_n^2 z^n$

$$\textit{we know that } \frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{(|a_n|)^2}$$

$$= \lim_{n\to\infty} \sup(|a_n|)^{2/n}$$

$$= \left(\lim_{n\to\infty} \sup|a_n|^{\frac{1}{n}}\right)^2 = (1/R)^2$$

$$\therefore \frac{1}{R_1} = \frac{1}{R^2} \Rightarrow R_1 = R^2$$

If $f(z) = \sum a_n z^n$ then what is the radius of convergence of $\sum n^3 a_n z^n$?

Solution:

Method:1

Let R be the radius of convergence of $f(z) = \sum a_n z^n$

$\because \frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{(a_n)}$ .... (1)

Let R, be the radius of convergence of $\sum n^3 a_n z^n$

$$\therefore \frac{1}{R_1} = \lim_{n\to\infty} \sup \sqrt[n]{|n^3 a_n|}$$

(i.e.) $\frac{1}{R_1} = \lim_{n\to\infty} \sup |n^3 a_n|^{1/n}$

we know that $|z_1 z_2| = |z_1||z_2|$

$$\therefore \frac{1}{R} = \lim_{h\to\infty} \sup |n^3||a_n|^{1/n}$$

$$= \left(\lim_{h\to\infty} \sup \left|n^{1/n}\right|^3\right)\left(\lim_{n\to\infty} \sup |a_n|^{1/n}\right)$$

$$w \cdot k \cdot t \lim_{n\to\infty} n^{1/n} = 1$$

$$\Rightarrow \frac{1}{R_1} = 1 \cdot \lim_{n\to\infty} \sup |a_n|^{1/n} = 1/R$$

$$\therefore R_1 = R$$

Hence radius of convergence of $\sum a_n z^n$ and $\sum n^3 a_n z^n$ are same.

Method: 2

$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ and

Here $a_n = n^3 a_n$

$$\Rightarrow \frac{1}{R_1} = \lim_{n\to\infty} \left|\frac{(n+1)^3 a_{n+1}}{n^3 a_n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{n^3(1+1/n)^3 a_{n+1}}{n^3}\right|$$

$$= \lim_{n\to\infty} \left|\left(1+\frac{1}{n}\right)^3\right| \cdot \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$= 1. \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$$\Longrightarrow \frac{1}{R_1} = \frac{1}{R}$$

$$\Rightarrow R_1 = \mathrm{R}$$

3. If $\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergent R_1 and R_2. Show that the radius of convergence of $\sum_n a_n b_n z^n$ is atleast $R_1 R_2$

Solution:

Let R_1 be the radius of convergence of $\sum a_n 2^n$ and R_2 be the radius of convergence of $\sum b_n z^n$

$\therefore \frac{1}{R_1} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$ and $\frac{1}{R_2} = \lim_{n\to\infty} \sup \sqrt{|\, b_n|}$ consider the series $\sum a_n b_n z^n$.

Let R be the radius of convergence of $\sum a_n b_n z^n$.

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|a_n b_n|}$$
$$= \lim_{n\to\infty} \sup |a_n|^{1/n} \cdot \lim_{n\to\infty} \sup |b_n|^{1/n}$$
$$\frac{1}{R} = \frac{1}{R_1} \frac{1}{R_2}$$
$$R = R_1 R_2 \text{ or } R \leqslant R_1 R_2$$

4.Expand $\frac{2z+3}{z+1}$ impower of $(z-1)$, what is its radius of convergence.

Solution:

Let $f(z) = \frac{2z+3}{z+1}$

put $h = z - 1 \Rightarrow z = h + 1$

$$f(z) = f(h+1) = \frac{2(h+1)+3}{(h+1)+1} = \frac{(2h+2)+3}{h+2} = \frac{2(h+1)+2+1}{(h+1)+1}$$
$$= \frac{2[h+1)+1]}{(h+1)+1} + \frac{1}{(h+1)+1} = \frac{2(h+1)+1]}{(h+1)+1} + \frac{1}{h+2}$$
$$= 2 + \frac{1}{h+2} = 2 + \frac{1}{2}\left(1 + \frac{h}{2}\right)^{-1}$$

$$= 2 + \frac{1}{2}\left(1 - \frac{h}{2} + \left(\frac{h}{2}\right)^2 \dots\dots..\right)$$

$$= 2 + \frac{1}{2} + \frac{1}{2}\left(-\frac{h}{2} + \left(\frac{h}{2}\right)^2 \dots\dots..\right)$$

∴ The radius of convergence is the large disc around 1 in which the function is analytic and -1 is the only point where the function is not analytic and the distance of -1 from 1 is 2. ∴ The required radius of convergence is R=2.

UNIT-II:

CAUCHY'S INTEGRAL FORMULA and LOCAL PROPERTIES OF ANALYTICAL FUNCTIONS: The Index of a point with respect to a closed curve – The Integral formula – Higher derivatives. Removable Singularities-Taylors's Theorem – Zeros and poles – The local Mapping – The Maximum Principle.

Chapter 2: Section 2: 2.1 to 2.7

2.1. The index of the point with respect to closed curve:

The index of the point (or) the winding point. The index of the point with respect to closed curve Gamma by the equation $n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$

Example:

We know that $\int_c \frac{dz}{z-a} = 2\pi i$

Where c is a circle with center a, hence $n(c, a) = 1$

Note: The bounded region is called the interior of c, other is called exterior of c.

Properties:

i) $n(\gamma, a)$ is an integer.

ii) $n(-\gamma, a) = -n(\gamma, a)$

iii) When a lies outside the circle then $n(\gamma, a) = 0$

iv) When γ is any closed curve then $n(\gamma, a)$ is constant is one region and zero in the un-bounded region. (or) $n(\gamma, a) = n(\gamma, b)$ if a and b ∈ same region determined by γ and $n(\gamma, a) = 0$ if a ∈unbounded region determined by γ.

Proof:

(i) We know that $n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a} = k$

Now k is an integer.

∴ $n(\gamma, a)$ is an integer

(ii) $n(-\gamma, a) = -n(\gamma, a)$

$$Let\, n(-\gamma, a) = \frac{1}{2\pi i}\int_{-\gamma}\frac{dz}{z-a}$$

$$= \frac{-1}{2\pi i}\int_{+\gamma}\frac{dz}{z-a} = -n(\gamma, a)$$

$$n(-\gamma, a) = -n(\gamma, a)$$

iii) Given a lies outside the circle $n(\gamma, a) = 0$

To prove that $n(\gamma, a) = 0$. Let γ lies inside of the circle

Then we get, $n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = 0$

$\Rightarrow n(\gamma, a) = 0$, ∀ point a outside of the circle.

iv) Let a and b be two points in the bounded region γ, such that γ does not passes through a and b.

$$n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}$$

$$n(\gamma, b) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-b}$$

$$n(\gamma, a) - n(\gamma, b) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} - \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-b}$$

$$= \frac{1}{2\pi i}\left[\int_{\gamma}\frac{dz}{z-a} - \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-b}\right]$$

$$= \frac{1}{2\pi i}[\log(z-a) - \log(z-b)]$$

$$n(\gamma, \mathrm{a}) - n(\gamma, b) = \frac{1}{2\pi i}\left[log\left[\frac{z-a}{z-b}\right]\right] \qquad \text{.............. (1)}$$

Since $\log\left(\frac{z-a}{z-b}\right)$ is never real and less than or equal to zero (≤ 0) but the index number should be positive ≥ 0

$$0 \leq \log\left[\frac{z-a}{z-b}\right] \leq 0$$
$$\Rightarrow \log\left[\frac{z-a}{z-b}\right] = 0$$
$$(1) \Rightarrow n(\gamma, a) - n(\gamma, b) = 0$$
$$\Rightarrow n(\gamma, a) = n(\gamma, b)$$

If $a \in$ unbounded region determined by γ.

$$\therefore n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{z-a}$$

when $|a|$ is sufficiently large and γ lies inside of the disc.

$$\therefore |z| \leq \gamma < |a|$$
$$\therefore n(\gamma, a) = 0$$

Note:

As a point set γ is closed, bounded and its complements in the extended plane is open.

$\Rightarrow$ Union of disjoint region.

Lemma :1

Let z_1, z_2 be two points an a closed Curve γ. which does not passes though the origin denote that sub arc is z_1 to z_2 in the direction of the curve by γ_1 and sub arc from z_2 to z_1 by γ_2 Suppose that z_1 lies the lower half plane and z_2 in the upper half plane. if γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis. Then $n(\gamma, 0) = 1$.

Proof:

draw the half line L_1, L_2 from the origion through z_1 and z_2.

Let ζ_1, ζ_2 be the point in which L_1 and L_2 intersect. The circle c about the origin

IF C is described in the positive sense then arc c_1 from ζ_1 and ζ_2 does not intersect the negative axis. and arc c_2 from ζ_2 and ζ_1 does not intersect the positive axis.

Let the direct line segment z_1 to ζ_1 and from z_2 to ζ_2 be denote by δ_1 and δ_2 respectividy.

introducing the closed curves σ_1 and σ_2 by mathematical simple symbol by positive sign for positive direction and negative sign for negative direction (or) opposite direction

$\therefore \sigma_1 = \gamma_1 + \nu_2 - c_1 - \delta_1$

& $\sigma_2 = \gamma_2 + \delta_1 - c_1 - \delta_2$

We find that $n(2,0) = n(c,0) + n(\sigma_1, 0) + n(\sigma_2, 0)$ ………..(1)

$\because$ cancellation is opposite direction.

Note that, σ_1 does not meet the negative axis.

Hence origin e unbounded region determined by σ_1 and hence we up obtain $n(\sigma_1, 0) = 0$ and for a similar reason $n(\sigma_2, 0) = 0. \Rightarrow\ n(\gamma, 0) = n(c, 0) + 0 + 0$

$\Rightarrow n(\gamma, 0) = n(c, 0) = 1 \Rightarrow n(\gamma, 0) = 1$

Problem:1

Compute $n(c, 0)$ where c is curve given by equation $z = z(t) = e^{i2\pi nt}, 0 \leq t \leq 1$.

Solution:

We know that, $n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ …………..(1)

To find: $n(c, 0)$.

$$\left.\begin{aligned} \gamma = c, a = 0, z = z(t) &= e^{i2\pi nt} \\ dz &= i2\pi n e^{i2\pi nt} dt \end{aligned}\right\} \text{ sub in (1)}$$

$$(1) \Rightarrow \; n(c,0)\frac{1}{2\pi i}\int_{t=0}^{1}\frac{i2\pi n e^{i2\pi nt}dt}{e^{i2\pi nt}}$$

$$\begin{aligned} n(c,0) &= \int_{t=0}^{1} n dt \\ &= n[t]_0^1 \\ &= n(1-0) \\ n(c,0) &= n. \end{aligned}$$

Problem:2

Find $n(\gamma, a),\; z = z(t) = e^{4\pi it}, 0 \le t \le 2$

Solution:

Here $a = 0, z(t) = z = e^{4\pi it}, 0 \le t \le 2$

$dz = 4\pi i e^{4\pi it} dt$

We know that, $n(\gamma, a) = \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a}$ …………..(1)

$(1) \Rightarrow \; n(\gamma, 0) = \frac{1}{2\pi i}\int_{t=0}^{2}\frac{4\pi i e^{4\pi it}dt}{e^{4\pi it}}$

$=2\int_{0}^{2} dt = 2[t]_0^2$

$n(\gamma, 0) = 4$

2.2. The Integral formula:

Theorem 1: The Cauchy's integral Theorem

Suppose that $f(z)$ is analytic is an open disc Δ and Let γ be a closed curve in Δ. For any point a not an γ. $n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)}dz$. Where $n(\gamma, a)$ is the index of a with respect to γ.

Proof:

Let $f(z)$ be analytic function in open disc Δ

Let Y be a closed curve Δ.

Let $a \in \Delta$ but a does not, lie an γ

consider the function $F(z) = \frac{f(z)-f(a)}{z-a}$

This function is analytic at all points except at $z = a$. does not pole whole

$$\therefore \lim_{x \to a}(z-a)F(z) = \lim_{z \to a}(z-a)\frac{f(z)-f(a)}{z-a}$$
$$= f(a) - f(a)$$

$$\Rightarrow \lim_{z \to a}(z-a)F(z) = 0$$

$\Rightarrow z = a$ is an exceptional point of $F(z)$

$\therefore$ By cauchy theorem [Let $F(z)$ be a function which is analytic inside and an a simple closed curve C. Then $\int_C f(z)dz = 0$.]

$$\therefore \int_\gamma F(z)dz = 0$$

(i.e.) $\int_\gamma \frac{f(z)-f(a)}{z-a} dz = 0.$

(i.e.) $\int_\gamma \frac{f(z)}{z-a} dz - f(a)\int_\gamma \frac{dz}{z-a} = 0$

(i.e.) $\int_\gamma \frac{f(z)}{z-a} dz = f(a) 2\pi i\, n(\gamma, a)$

(i.e.) $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a} dz = f(a) \cdot n(\gamma, a)$

When $n(\gamma, a) = 1$

we get, $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a} dz$

Problem 3:

compute $\int_{|z|=2} \frac{dz}{z^2+1}$

Solution:

By Cauchy's Integral Formula,

$h(\gamma, a)\, f(a) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-a} dz$ …………… (1)

$f(z) = 1, \;\Rightarrow\; z - a = z^2 + 1 = (z+i)(z-i)$

$= z^2 - i^2 = z^2 + 1$

$$\Rightarrow \int \frac{dz}{(z+i)(z-i)} = \int \frac{A}{z+i} dz + \int \frac{B}{z-i} dz$$

$$\Rightarrow \frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i}$$

$$\frac{1}{(z+i)(z-i)} = \frac{A(z-i) + B(z+i)}{(z+i)(z-i)}$$

$1 = A(z-i) + B(z+i)$

put $z =$ i

$1 = 0 + 2$ Bi

$$\Rightarrow B = \frac{1}{2i}$$

Put $z = -i \Rightarrow 1 = A(-2i) + 0 \Rightarrow 1 = -2Ai$

$\Rightarrow A = -1/2i.$

$$\frac{1}{z^2+1} = -\frac{1}{2i}\frac{1}{(z+i)} + \frac{1}{2i}\int \frac{1}{(z-i)}$$

$$\therefore \int \frac{dz}{z^2+1} = \frac{-1}{2i}\int_{|z|=2} \frac{dz}{z+i} + \frac{1}{2i}\int_{|z|=2} \frac{dz}{z-i} \qquad \text{………..} (2)$$

$f(z) = 1,\ z + i = z - (-i)$

$\Rightarrow a = -i$

$\therefore a = -i$ lies inside $c: |z| = 2$

$\therefore n(c, -i) = 1$

$f(a) = f(-i) = 1$

$\therefore$ By Cauchy Integral Formula,

$$= \frac{-1}{2\pi i}\left(\frac{1}{2i}\right)\int_{|z|=2} \frac{dz}{z-(-i)}$$

$$-4\pi i^2 = \int_{|z|=2} \frac{dz}{z-(-i)}$$

$$\Rightarrow 4\pi = \int_{|z|=2} \frac{dz}{z-(-i)} \qquad \text{…………} (3)$$

$$a = i \Rightarrow -4\pi = \int_{|z|=2} \frac{dz}{z-i} \qquad \text{………..} (4)$$

Sub equation (3) (4) in (2).

$$\int_{|z|=2} \frac{dz}{z^2+1} = 4\pi - 4\pi$$

$$\Rightarrow \int_{|z|=2} \frac{dz}{z^2+1} = 0.$$

Problem 4:

compute $\int_{|z|=1} \frac{\cos z}{z(z-4)} dz$

Solution:

By Cauchy's Integral formula,

$$n(\gamma,a)f(a)=\frac{1}{2\pi i}\int\frac{f(z)}{z-a}dz$$

$$f(z)=\cos z, z-a=z(z-4)$$

$$\frac{a}{2}=0,4.$$

$$\frac{1}{(z-0)(z-4)}=\frac{A}{z-0}+\frac{B}{z-4}$$

$$\frac{1}{(z-0)(z-4)}=\frac{B(z-4)+B(z-0)}{(z-0)(z-4)}$$

put $z=4$

$1=0+4B$

$\Rightarrow B=1/4$

$z=0$

$1=4A$

$\Rightarrow A=-1/4$

$$n(c,0)f(0)=\frac{1}{2\pi i}\left[\int_{|z|=1}-\frac{1}{4}\frac{\cos z}{z}dz+1/4\int\frac{\cos z}{z-4}dz\right]\quad \ldots\ldots\ldots\ldots(2)$$

$\Rightarrow n(c,0)=0$

$f(0)=\cos 0=1$

$$(1)=\frac{1}{2\pi i}-\frac{1}{4}\int_{|z|=1}\frac{\cos z}{z}dz$$

$$-8\pi i=\int\frac{\cos z}{z}dz$$

Similarly, $\Rightarrow\int\frac{\cos z}{(z-4)}dz=8\pi i$

from (3), (4) Sub in (2)

$$\Rightarrow \int \frac{\cos 2}{z(z-1)}dz = 8\pi i - 8\pi i = 0$$

$$\Rightarrow \int \frac{\cos z}{(z-4)}dz = 0.$$

Problem 5:

What is the value of $\int_c \frac{dz}{z-a}$ if c is a circe and a lies outside c.

Solution:

Given a lies out side c.

$\therefore n(c,a) = 0$

We know that, By Cauchy Integral formula.

$$n(\gamma,a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{df(z)}{z-a}dz$$

from given, $f(z) = 1$.

$$n(c,a) = 0$$
$$f(\alpha) = 1$$
$$\therefore \frac{1}{2\pi i}\int \frac{dz}{z-a}$$
$$\Rightarrow \int_\gamma \frac{dz}{z-a} = 0$$

Problem 6:

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}, \ |a| \neq \rho.$$

Solution:

given $|z| = \rho$

$|z|^2 = e^2 \Rightarrow z\bar{z} = \rho^2$

Let $z = \rho e^{i\theta}$

$dz = \rho e^{i\theta} \cdot id\theta \Rightarrow d\theta = \dfrac{dz}{iz}$

$|dz| = |\rho e^{i\theta} id\theta|$

$= |\rho|\left|e^{i\theta}\right||i||d\theta|$

$= z(1)(1)|d\theta|$

$\Rightarrow |dz| = \rho \dfrac{dz}{iz}$

$|dz| = -i\rho \dfrac{dz}{z}.$

$|z-a|^2 = (z-a)(\overline{z-a})$

$= (z-a)(\bar{z}-\bar{a})$

$|z-a|^2 = (z-a)\left(\dfrac{\rho^2}{z} - \bar{a}\right)$

$$\therefore \int_{|z|=\rho} \frac{-i\rho \frac{dz}{z}}{(z-a)\left(\frac{\rho^2}{z} - \bar{a}\right)} = \int_{|z|=e} \frac{-i\rho dz}{z(z-a)\left(\frac{\rho^2 - \bar{a}z}{z}\right)}$$

$\int_{|z|=\rho} \frac{-i\rho dz}{(z-a)(\rho^2 - \bar{a}z)}$ ………… (1)

We know that,

The Cauchy's Integral Formula,

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$$

$$\Rightarrow = 2\pi i n(\gamma, a)f(a) = \int_C \frac{f(z)}{z-a}dz \quad \ldots\ldots\ldots\ldots.(2)$$

From (1),

Let, $f(z) = \frac{-ie}{\rho^2 - \bar{a}z}$.

$$\Rightarrow f(a) = \frac{-ie}{\rho^2 - \bar{a} \cdot a} = \frac{-ie}{\rho^2 - a^2}$$

$n(c: |z| = \rho, a) = 1.$ Sub in (2),

$$\Rightarrow \frac{2\pi\rho}{\rho^2 - a^2} = \int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

$$\Rightarrow \int_{|z|=\rho} \frac{|\,dz1}{|z-a|^2} = \frac{2\pi\rho}{\rho^2 - a^2}$$

Problem 7:

$$\int_{|z|=1} |z-1||dz|$$

Solution:

given c: $|z| = 1$ $\quad |z| = \gamma$, $z = re^{i\theta}$.

$$z = (1)e^{i\theta}$$
$$dz = e^{i\theta} \cdot id\theta$$
$$|dz| = \left|e^{i\theta}\right||i||d\theta| = |1||1|d\theta\big)$$

$$\left|e^{i\theta}\right| = |\cos\theta + i\sin\theta| \;\Rightarrow |dz| = |d\theta|$$
$$= \sqrt{\cos^2\theta + \sin^2\theta} \qquad 0 \le \theta \le 2\pi$$
$$= 1$$
$$|i| = \sqrt{0+1} = \sqrt{i} = 1$$
$$(z-1)^2 = (z-1)(\overline{z-1})$$
$$= (z-1)(z-1)$$
$$= \left(e^{i\theta} - 1\right)\left(e^{-i\theta} - 1\right)$$
$$= \left(e^{i\theta}e^{-i\theta} - e^{i\theta} - e^{-i\theta} + 1\right)$$
$$= (1 - \cos\theta - i\sin\theta - \cos\theta + i\sin\theta + 1)$$
$$= (2 - 2\cos\theta)$$
$$= 2(1 - \cos\theta) = 2[2\sin^2\theta/2]$$

$$|z-1|^2 = 4\sin^2 \theta/2$$

$$\Rightarrow |z-1| = 2\sin \theta/2$$

$$\int_{|z|=1} |z-1||dz| = 2\int_0^{2\pi} \sin \theta/2 d\theta$$

$$= 2\left[\frac{-\cos \theta/2}{1/2}\right]_0^{2\pi}$$

$$= -4[\cos \theta/2]_0^{2\pi}$$

$$= -4[\cos \pi - \cos 0]$$

$$= -4[-1-17$$

$$= -4[-2] = 8$$

2.3. Higher derivatives:

Theorem: 1 (Cauchy's representation formula for the derivative)

An analytic function $f(z)$ in a region as derivatives of all orders which are also analytic in the same region Ω .

Proof:

We know that, Cauchy's Integral formula,

$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ ……………. (1)

Choose, $|\Delta z|$ is small.

Such that $f'(z) = \lim\limits_{\Delta z \to 0} \frac{f(z+\Delta\Delta - f(x)}{\Delta z}$

$z + \Delta z$ lies between ν.

Now, $f(z+\Delta z) - f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-(z+\Delta z)} d\zeta$

$$= \frac{1}{2\pi i}\left[\int_\gamma \frac{f(\zeta)}{\zeta - z - \Delta z} d\zeta - \int_\nu \frac{f(\zeta)}{\zeta - z} d\zeta\right]$$

$$= \frac{1}{2\pi i}\left[\int_\gamma f(\zeta)\left[\frac{1}{\zeta - z - \Delta z} - \frac{1}{\zeta - z}\right] d\zeta\right]$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{\zeta - z - z + z + \Delta z}{(\zeta - z - \Delta z)(\zeta - z)}\right] d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{\Delta z}{(\zeta - z)(\zeta - z - \Delta z)}\right] d\zeta$$

$(\div)\Delta z$ on both sides,

$$\therefore \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)(\zeta - z - \Delta z)} d\zeta$$

Sub, $\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - 2)^2} d\zeta$ on both sides,

$$\Rightarrow \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{1}{(\zeta - z)(\zeta - z - \Delta z)} - \frac{1}{(\zeta - z)^2}\right] d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{\zeta - \mathrm{z} - \zeta + z + \Delta z}{(\zeta - z)^2(\zeta - z - \Delta z)}\right] d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\left[\frac{\Delta z}{(\zeta - z)^2(\zeta - z - \Delta z)}\right] d\zeta$$

$$= \frac{\Delta z}{2\pi i}\int_\gamma \frac{f(\xi)}{(\zeta - z)^2(\zeta - z - \Delta z)} d\zeta$$

Taking modules on both sides,

$$\therefore \left| \frac{f(z+\Delta z - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta\right|$$

$$= \left|\frac{\Delta z}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2(\zeta - z - \Delta z)} d\zeta\right|$$

$$\leq \frac{|\Delta z|}{2\pi}\int \frac{|f(\zeta)|}{|\zeta - z|^2|\zeta - z - \Delta z|} d\zeta \qquad \ldots\ldots\ldots\ldots (2)$$

Let $M =$ max of $f(4)$

Let $\delta =$ the mean distance of points ζ, on γ from z

Since $f(\zeta)$ is an analytic an γ.

He should be continuous on γ.

$$\Rightarrow |f(\zeta)| \leq M \; on \; \gamma \quad \ldots\ldots\ldots\ldots (3)$$
$$|\zeta - z| > \delta$$
$$\Rightarrow \frac{1}{|\zeta - z|} \leq \frac{1}{\delta}$$
$$\Rightarrow \frac{1}{|\zeta - z|^2} \leq \frac{1}{\delta^2} \quad \ldots\ldots\ldots\ldots\ldots (4)$$
$$\geq |\zeta - z - \Delta z| = |(\zeta - z) - \Delta z|$$
$$\geq |(\zeta - z)| - |\Delta z| \geq \delta - |\Delta z|$$

$$\Rightarrow \frac{1}{|\zeta - z - \Delta z|} \leq \frac{1}{\delta - |\Delta z|} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We know that, $\int_\gamma |dr| = l =$ length of a curve γ ……………..(6)

Sub (3), (4), (5) & (6) in (2).

$$\therefore \frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi}\int \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \Big| \leq \frac{|\Delta z|}{2\pi} \frac{ml}{\delta^2(\delta - |\Delta z|)}$$
$$\rightarrow 0 \text{ as } \Delta z \rightarrow 0$$

$$\therefore \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right] \rightarrow 0 \text{ as } \Delta z \rightarrow 0$$

(i.e.,) $f'(z) - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^2} = 0$

1st derivative

$$\Rightarrow f'(z) = \frac{1!}{2\pi i}\int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

$$f''(z) = \frac{2!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^3}$$

$\dots\dots\dots\dots\dots..$

$$f^n(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

Evaluate 1:

$\int_{|z-1|=2} \frac{\sin z}{(z-1)^2} dz.$

Solution:

We know that, $f^n(a) = \frac{n!}{2\pi i} \cdot \int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}$ …………(1)

$$\Rightarrow \frac{2\pi i}{n!} f^n(a) = \int_\gamma \frac{f(z)dz}{(z-a)^{n+1}}$$

$n + 1 = 2 \Rightarrow n = 1$

$C: |z - 1| = 2 \qquad (\because c: |z - a| = r)$

$a = 1$

$f(z) = \sin z$

$f'(z) = \cos z$

$f'(1) = \cos 1 = f'(a)$

$\therefore 0 \Rightarrow \frac{f'(a)2\pi i}{1!} = \int_\gamma \frac{\sin z dz}{(z-1)^2}$

$$= \frac{(\cos 1)(2\pi i)}{1!} = \int_\gamma \frac{\sin z dz}{(z-1)^2}$$

$$\Rightarrow \int_\gamma \frac{\sin z dz}{(z-1)^2} = 2\pi i(\cos 1)$$

Lemma 1:

Suppose that $\phi(\zeta)$ is continuous an arc ν. Then the function $F_n(z) = \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta-2)^n}$ is analytic $F_n(z)$. In each of the region determined by ν and the its derivatives is $F_n'(z) = nF_{n+1}(z)$.

Proof:

To prove that $F(z)$ is continuous

Let z_0 be a point not an ν. and Choose neighbourhood $|z| - z_0 |< 8$.

So that it does not meet α.

By restricting z to the smaller neighbor $|z - z_0| < \delta/2$.

$\therefore$ we find that $|\zeta - z| > \delta/2\ \forall t \in \gamma$

Now,

$$F_1(z) - F_1(z_0) = \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)} - \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)}$$

$$= \int_\gamma \phi(\zeta)\left[\frac{1}{(\zeta - z)} - \frac{1}{(\zeta - z_0)}\right] ds$$

$$= \int_\gamma \phi(z)\left[\frac{\zeta - z_0 - \zeta + z}{(\zeta - z)(\zeta - z_0)}\right] ds$$

Taking modulus on both sides,

$$|F_1(z) - F_1(z_0)| = |\int_\gamma \phi(z)\left[\frac{z-z_0}{(\zeta-z)(\zeta-z_0)}\right] d\zeta|$$

$$|F_1(z) - F_1(z_0)| \le |z - z_0| \int_\gamma \frac{|\phi(\zeta)||d(\zeta)|}{|\zeta-z||\zeta-z_0|} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

given $\phi(\zeta)$ is continuous $\Rightarrow |\phi(\zeta)| \le M$ …… (2)

$$\& \ |\zeta - z| = \frac{\delta}{2}$$

$$\Rightarrow \frac{1}{|\zeta - z|} < \frac{2}{\delta} \quad \ldots\ldots\ldots .(3)$$

and Let $\int_{\gamma} |d\zeta| = l$ ....(4)

$l =$ length of the are γ.

Already we know that,

$$|\zeta - z_0| > \delta$$
$$\Rightarrow \frac{1}{|\zeta - z_0|} < \frac{1}{\delta} \ldots\ldots\ldots\ldots (5)$$

Sub (2)(3)(4) & (5) in (1).

$$\therefore (1) \Rightarrow$$
$$|F_1(z) + F_1(z_0)| \leqslant 2|z - z_0| \frac{ml}{\delta^2} < \varepsilon$$
$$\text{Take, } \eta = \frac{\varepsilon\delta^2}{2ml}.$$
$$\Rightarrow \frac{\eta_{2ml}}{\delta^2} = \varepsilon$$

$\Rightarrow |F_1(z) - F_1(z_0)| < \varepsilon$

$\Rightarrow F_1(z)$ is continuous of z_0.

From (1) we get,

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(z)$$

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_{\gamma} \frac{\phi(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)}$$
$$\Rightarrow \lim_{z \to z_0} \frac{F_1(z) - F_1(z_0)}{z - z_0} = \lim_{z \to z_0} \int_{\gamma} \frac{\phi(\zeta) d\tau}{(\zeta - z)(\zeta - z_0)}$$
$$\Rightarrow F'(z_0) = \int_{\gamma} \frac{\phi(\zeta) d\zeta}{(\zeta - z_0)(\zeta - z_0)}$$
$$= \int_{\gamma} \frac{\phi(\zeta) d\zeta}{(\zeta - z_0)^2}$$
$$F_1'(z_0) = F_2(z_0)$$

The General case by induction on n. we have already proved for $n = 1$.

We shall assume that the result is true for $n = (n-1)$.

(i.e.,) We assume that $F'_{n-1}(z) \overset{(n-1)}{=} F_{n-1}(z)$

To prove that: $(2) \cdot F'_n(z) = nF_{n+1}(z)$.

$$F_n(z) - F_n(z_0) = \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n} - \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)^n}$$

$$= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} \cdot \frac{(\zeta - z_0)}{(\zeta - z)} d\zeta = -\int_\gamma \frac{\phi(\zeta)dz}{(\zeta - z_0)^n}$$

$$= \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{n-1}(\zeta - z_0)} \left\{\frac{(\zeta - z) + (z - z_0)}{(\zeta - z)}\right\} d\zeta - \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)^n}.$$

$$F_n(z) - F_n(z_0) = \int_\gamma \frac{(\zeta - z)\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)} + \int_\gamma \frac{(z - z_0)\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)} - \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)^n}$$

$$= \int_\gamma \frac{(\zeta - z)\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)} - \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z_0)^n} + \int_\gamma \frac{(z - z_0)\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

$$= \int_\gamma \frac{[(\zeta - z)(\zeta - z_0)^{n-1} - (\zeta - z)^n]}{(\zeta - z)^n(\zeta - z_0)^n} \phi(\zeta)d\zeta + (z - z_0)\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

$$= \int_\gamma \frac{(\zeta - z)[(\zeta - z_0)^{n-1} - (\zeta - z)^{n-1}]}{(\zeta - z)^n(\zeta - z_0)^n} \phi(\zeta)d\zeta + (z - z_0)\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)}.$$

$$= \int_\gamma \frac{(\zeta - z_0)^{n-1} - (\zeta - z)^{n-1}}{(\zeta - z)^{n-1}(\zeta - z_0)^n} \phi(\zeta)d\zeta + (z - z_0)\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

$[\ a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

$(\zeta - z_0)^{n-1} - (\zeta - z)^{n-1} = (\zeta - z_0)(\zeta - z)[(\zeta - z_0)^{n-2} + (\zeta - z_0)^{n-3}(\zeta - z).$

$+(\zeta - z_0)^{n-1} + (\zeta - z)^2_+ \dots (\zeta - z)^{n-2}]$

$$= \frac{\int_\gamma (\zeta - z_0)[(\zeta - z_0)^{n-2} + (\zeta - z_0)^{n-3}(\zeta - z) + (\zeta - z_0)^{n-1}(\zeta - z)^2_+ \dots (\zeta - z)^{n-2}]\phi(\zeta)d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)^n} + (z - z_0)\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta - z)^n(\zeta - z_0)}$$

$$\frac{F_n(z)-F_n(z_0)}{z-z_0} = \frac{\int_\gamma [(\zeta-z_0)^{n-2}+(\zeta-z_0)^{n-3}(\zeta-z)+\cdots(\zeta-z)^{n-2}]\phi(\zeta)d\zeta}{(\zeta-z)^{n-1}(\zeta-z_0)^n}+\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta-z)^n(\zeta-z_0)}$$

getting lim $z \to z_0$. Both side we get,

$$F_n'(z_0) = \int_\gamma \frac{(\zeta-z_0)^{n-2}+(\zeta-z_0)^{n-2}+\cdots(n-1)\text{ time }\phi\zeta d\zeta}{(\zeta-z_0)^{2n-1}} + \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta-z)^{n+1}}$$

$$\int_\gamma \frac{(n-1)\,(\zeta-z_0)^{n-2}}{(\zeta-z_0)^{2n-1}}\phi(\zeta)d\zeta + \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta-z)^{n+1}}$$

$$= (n-1)\int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta-z_0)^{n+1}} + \int_\gamma \frac{\phi(\zeta)d\zeta}{(\zeta-z_0)^{n+1}} [\because z_0 = z]$$

$$= (n-1)F_{n+1}(z_0) + F_{n+1}(z_0)$$

$$\Rightarrow F_n'(z) = (n-1+1)\,F_{n+1}(z).$$

$$F_n'(z) = nF_{n+1}(z)$$

Theorem 2: (Morera's Theorem (or) Converse part of Cauchy s Theorem)

If $f(z)$ is defined and cts in region Ω. and if $\int_\gamma f(z)dz = 0, \forall$ closed curves γ in Ω.Then $f(z)$ is analytic in Ω analytic in Ω.

Proof:

Let us closed curve γ in the region Ω.

By hypothesis

$$\Rightarrow \int_{AMB}^{AMBNA} f(z)dz + \int_{\text{ANA}} f(z)dz = 0$$

$$\int_{\text{AMB}} f(z)dz = -\int_{\text{BNA}} f(z)dz$$

$$= \int_{ANB} f(z)dz$$

$\therefore$ The value of the integral $\int_\gamma f(z)dz$ independent of path joining A to B

Let z_0 represent A and z represent Let us choose the straight line segment joining A to B.

$\because$ we write. $\int_{z_0}^{z} f(z)dz = F(z)$ ………… (1)

Now, $F(z + \Delta z) = \int_{z_0}^{z+\Delta z} f(z)dz$ …………(2)

Such that.

$z + \Delta z$ lies in side the region Ω.

$(2) - (1) \Rightarrow F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(z)dz - \int_{z_0}^{z} f(z)dz$

$$F(z + \Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(z)dz + \int_{z}^{z_0} f(z)dz$$
$$= \int_{z}^{z+\Delta z} f(\zeta)d\zeta$$

$\therefore$ (2) - (1) $\Rightarrow$

$\therefore$ ΔZ on both side

$$\frac{F(z + \Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)d\zeta$$

$$\Rightarrow \frac{F(z + \Delta \Rightarrow -F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)d\zeta - f(z)$$
$$= \frac{1}{\Delta z}\int_{z}^{z1\Delta z} f(\zeta)d\zeta - \frac{\Delta z}{\Delta z}f(z)$$
$$= \frac{1}{\Delta z}\int_{2}^{z+\Delta z} f(\zeta)d\zeta - \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)d\zeta$$
$$= \int_{z}^{z+\Delta z} \left[\frac{f(\zeta) - f(z)}{\Delta z}\right] d\zeta$$

Taking modulus on both sides,

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| = \left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z}[f(\zeta)-f(z)]d\zeta\right|$$

$$\leq \frac{1}{\Delta z}\int_{z}^{z+\Delta z}|f(\zeta)-f(z)|d\zeta \quad \because |z-\zeta|<\delta \Rightarrow |f(z)-f(\zeta)|<\varepsilon]$$

$$\leq \frac{\varepsilon}{\Delta z}\int_{z}^{z+\Delta z}d\zeta$$

$$< \frac{\varepsilon}{\Delta z}[\zeta]_{z}^{z+\Delta z}$$

$$< \frac{\varepsilon}{\Delta z}[z+\Delta z-z]$$

$$< \frac{\varepsilon}{\Delta z}(\Delta z)$$

$$< \varepsilon$$

$$\Rightarrow \left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| < \varepsilon$$

Let $\Delta z \to 0$ $\lim\limits_{\Delta z\to 0}\dfrac{F(z+\Delta z)-F(z)}{\Delta z} - \lim\limits_{\Delta z\to 0} f(z) \leq \varepsilon$

$\Rightarrow F'(z)-f(z)=0$ as $\Delta z \to 0$

$\Rightarrow F'(z)=f(z)$

$F(z)$ is analytic

$\Rightarrow F''(z)=f'(z)$

$\Rightarrow f(z)$ is analytic

Theorem 2:

Cauchy's inequality (or) Cauchy's Estimate Theorem

If $f(z)$ is analytic with in and an a circle c. given by $|\ z-al=r$. lies inside $\triangle$ and if $|f(z)| \leqslant M$ for any z on C. Then $|f^n(a)| \leq \frac{n!\,m}{r^n}$.

Proof:

We know that Cauchy's higher derivative,

$$f^n(z) = \frac{n!}{2\pi i}\int_c \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

$\therefore$ Put $z = a$ & $y = z$,

$$\Rightarrow f^n(a) = \frac{n!}{2\pi i}\int_z \frac{f(z)dz}{(z-a)^{n+1}}$$

$$\Rightarrow |f^n(a)| = \left|\frac{n!}{2\pi i}\int_c \frac{f(z)dz}{(z-a)^{n+1}}\right|, \quad \because |i| = 1$$

$$\leq \frac{n!}{2\pi i}\int_c \frac{|f(z)|dz}{|z-a|^{n+1}}$$

$$= \frac{n!}{2\pi}\frac{m}{\gamma^{n+1}}\left[\int_c dz\right]$$

$[\because$ given $|f(z)| \leqslant m$

$$= \frac{n!}{2\pi}\frac{m}{r^{n+1}}[2\pi r]$$

$$= \frac{n! \cdot m \cdot \gamma}{\gamma^{n+1}}$$

$$= \frac{n!\,m}{r^n}$$

$$\therefore |f^n(a)| < \frac{n!\,m}{r^n}$$

Entire Function:

A function which is analytic in every finite region of the complex plane is called an Entire function.

Example :- $f(z) = e^z$.

Theorem: 3 (Liouville's Theorem)

A function which is analytic and bounded in the whole plane must reduce to a constant (or)

If a function $f(z)$ is analytic for all finite value of z and is bounded then $f(z)$ is a constant.

Proof:

Let a be any point of the plane.

Let c be the circle with Centre at a and radius r.

We know that,

Cauchy's Integral formula, $f'(a) = \frac{1!}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^2}dz$ ....(1)

Given, $f(z)$ is bounded fz.

(i.e.,) $|f(z)| \leq M, \forall z.$ Taking modulus for equation (1) on both sides,

$$\therefore |f'(a)| = \left|\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{(z-a)^2}dz\right|$$
$$< \frac{1}{2\pi}\int_{\gamma}\frac{|f(z)1}{|(z-a)^2|}|dz|$$
$$< \frac{1}{2\pi}\int_{2}\frac{M}{|z-a)^2}dz$$
$$= \frac{m}{2\pi r^2}\int_{\gamma} dz \;= \frac{m}{2\pi r^2}(2\pi r)$$
$$\Rightarrow |f'(a)| < \frac{m}{r} \quad \dots\dots\dots\dots(2)$$

This is true for any circle with radius r.

We know that, the complex plane is a circle with infinite radius.

Hence we can take $\lim_{r\to\infty}$ on both sides,

(2) $\Rightarrow \lim_{r\to\infty}|f'(a)| < \lim_{r\to\infty}\frac{M}{r} \to 0.$

(i.e.,) $f'(a) = 0.$ $\forall$ point in the z-plane

$$\Rightarrow \; f'(z) = 0.$$
$$\Rightarrow f(z) = \text{ constant}$$

Corollary: 1 (Fundamental Theorem of Algebra)

If $P(z)$ is polynomial of degree n, $n \geq 1$. With real and complex coefficient. Then the equation $p(z) = 0$ as at least one root. (or)

Every polynomial in z of degree $n \geq 0$. must have at least one zero.

Proof:

Let $f(z)$ be a non constant polynomial.

(i.e.,) Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ where $a_n \neq 0$.

Suppose that $f(z)$ is non zero (or) never zero in the whole plane then we can say that $f(z)$ is analytic in the whole complex plane and $\therefore \frac{1}{f(z)}$.

Since $f(z) \neq 0$.

$$\therefore |f(z)| = |a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n|$$
$$= |a_n z^n| \left| 1 + \frac{a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^n}{a_n z^n} \right|$$
$$\to 0 \text{ as } |z| \to \infty$$
$$\frac{1}{|f(z)|} \to 0 \text{ as } |z| \to \infty$$

$\therefore \frac{1}{|f(z)|}$ is bounded in the wholeplare. (ie) $\frac{1}{f(z)}$ is analytic and bounded in the whole complex plane. $\therefore$ Liouville's theorem, $\frac{1}{f(z)}$ is constant.

$\Rightarrow f(z)$ is constant. Which is a contradiction

$\therefore$ our assumption that $f(z) \neq 0$ is wrong. $\therefore$ $f(z)$ as atleast one zero.

2.4. Local Properties of Analytic Function:

Definition: Singularities

A point $z = a$ is said to be a singularity of the function $f(z)$ if $f(z)z$ is not analytic at

$z = a$.

Types of singularities:

1. Removable Singularity

2. Isolated Singularity

3. Poles singularity

4. Essential singularity

1.Removable singularity

Let $f(z)$ be an analytic function in a region except at $z = a$ and if

$\lim_{z\to a}(z-a)f(z) = 0$. Then $z = a$ is said to be Removable singularity of $f(z)$.

Theorem 1:

Suppose that $f(z)$ is analytic in a region Ω' obtain by omiting a point a from the region Ω. (i.e.,) $\Omega' = \Omega - \{a\}$. A necessary and sufficient condition that their exist for analytic function Ω. which co-inside with $f(z)$ in Ω is that $\lim_{z\to a}(z-a)f(z) = 0$ (or)

Necessary & Sufficient condition that their exist a unique analytic function $F(z)$ in Ω. That $f(z)$ in Ω is that $z = a$ is removable singularity and of $f(z)$.

(i.e.,) $\lim_{z\to a} f(z-a)f(z) = 0$.

Their exist unique analytic function $F(z)$ in Ω Such that $F(z) = f(z), \forall z \in \Omega'$. iff $z = a$ is a removable singularity of $f(z)$.

Proof:

Necessary part:

Given their exist an analytic function in Ω and which same as $f(z)$ in Ω'.

To prove that : $z = a$ is a removable singularity of $f(z)$.

(i.e.,) To prove that $\lim_{z\to a}(z-a)f(z) = 0$.

Given $F(\mathrm{z})$ is a analytic in Ω and $F(z)$ is also analytic at a.

$\Rightarrow F(z)$ is continuous at $z = a$.

(i.e.,) $\lim_{z\to a} F(z) = F(a)$.

(i.e.,) Given $\varepsilon > 0$. Their exist $\delta > 0$.

Such that, $|F(z) - F(a)| < \varepsilon$

When ever $0 < |z - a| < \delta$.

Now, $z \neq a$

$\Rightarrow |f(z) - F(a)| < \varepsilon$ …………… (1)

Whenever $0 < |z - a| < \delta$. and this $z \in \Omega'$.

$$(i.e.)\lim_{z\to a} f(z) = F(a)$$
$$\therefore \lim_{z\to a}(z-a)F(z) = 0, f(a)$$
$$\Rightarrow \lim_{z\to a}(z-a)f(z) = 0.$$

$\therefore z = a$ is the removable singularity of $f(z)$.

Sufficient part:

Given $z = a$ is a removable

Singularity of $f(z)$

$\Rightarrow z = a$ is singularity and exceptional point of $f(z)$ with

$\lim_{z\to a}(z-a)f(z) = 0$

To Prove that: their exist a unique analytic function $F(Z)$ in Ω.

(i.e.,) To show that $F(z) = f(z)\ \forall z \in \Omega'$

Its sufficient to prove that $F(z)$ is analytic at $z = a$

Let c be a circle about a.

Now, $f'(z)$ is analytic on Ω' and
c be a circle in Ω.

$\therefore n(c,z) = 1 \; \forall z \neq a.$

Thus a condition of Cauchy formula Satisfied.

$\therefore f(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)d\zeta}{(\zeta-z)}, \quad \forall \, z \neq a$ in ' c ' & analytic in Ω'. ……… (2)

$\Rightarrow f(z)$ is analytic in Ω' and $z \neq a$ in c

$\therefore F(z)$ is also analytic in c. where $c \in \Omega'$.

$\therefore f(z)$ is continuous an c.

$\therefore \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)}$ is analytic in any region determind by c.

$\therefore\Rightarrow \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)}$ is analytic in every point does not lie in C.

In particularly, $\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)}$ is analytic on Ω for $z = a$.

$\therefore \frac{1}{2\pi i}\int_c \frac{f(\zeta)d\zeta}{(\zeta-z)}$ is analytic in Ω for $z = a$. ……….. (3)

From (2) & (3).

Let us define in New function $F \rightarrow$.

$$F(z) = \begin{cases} \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)}, \forall \, z \neq a \text{ in analytic in } \Omega' \\ \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta - z)}, \forall z = a \text{ in analytic in } \Omega' \end{cases}$$

$\therefore F(z) = f(z) \; \forall z \in \Omega'$. where $z \neq a$

(i.e.) $F(z) = f(z), \forall z \in \Omega'$.

To prove that the unique of $F(z)$

Let $F(z)$ and $G(z)$ be two function satisfying the condition of theorem.

$\therefore$ $F(z)$ and $G(z)$ are analytic

$$FF(z) = f(z), \forall z \in \Omega'$$
$$\& \, G(z) = f(z), \forall z \in \Omega'$$
$$\Rightarrow F(z) - f(z) = 0$$
$$\& G(z) - f(z) = 0, \forall z \in \Omega'$$
$$\Rightarrow F(z) - f(z) = G(z) - f(z), \forall z \in \Omega'$$
$$\Rightarrow F(z) = G(z), \forall z \in \Omega'$$

Problem 1:

Prove that $Z = a$ is a removable singularity of $f(z) = \frac{\sin(z-a)}{z-a}$

Proof:

$f(z) \; = \frac{\sin(z-a)}{z-a}$ ………….. (1)

put $z \; = a$ in (1)

$$(1) \; \Rightarrow f(z) \; = \frac{\sin 0}{0} = \infty$$

$\Rightarrow$ $f(z)$ is not analytic at $z = a$.

$\Rightarrow$ $z = a$ is a a singularity of $f(z)$

$$\lim_{z \to a}(z-a)f(z) = \lim_{z \to a}(z-a)\frac{\sin(z-a)}{(z-a)}$$
$$= \sin 0 = 0.$$
$$\Rightarrow \lim_{z \to a}(z-a)f(z) = 0$$

$\therefore z = a$ in a Removable singularity of fz)

Problem 2:

Prove that:- $z = 0$ is a Removable Singularity of $f(z) = \frac{z - \sin z}{z^3}$.

Proof:

$f(z) = \frac{z - \sin z}{z^3}$ ………….. (1)

Put $z = 0$ in (1)

$(1) \Rightarrow f(2) = \frac{0-0}{0} = \infty$

$\Rightarrow f(z)$ is not analytic at $z = 0$

$\Rightarrow z = 0$ is singularity for $f(z)$ ……………(2)

$$\therefore \lim_{z\to a}(z-a)f(z) = \lim_{z\to 0}(z-\mathrm{a})\frac{z-\sin z}{z^3}$$

$$= \lim_{z\to 0}\frac{z-\sin z}{z^2} = \infty$$

$$f(z) = \frac{z-\sin z}{z^3} = \frac{1}{z^3}\left[z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]\right]$$

$$= \frac{1}{z^3}\left[z - z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$$

$$= \frac{1}{z^3}\left[\frac{z^3}{3!} - \frac{z^5}{5!}\right]$$

$$= \frac{z^3}{z^3}\left[\frac{1}{3!} - \frac{z^5}{5!}\right]$$

$$= \frac{1}{3!} - \frac{z^5}{5!}$$

Put $z = 0$,

$$f(0) = \frac{1}{3!} = \frac{1}{6}$$

$$\therefore f(z) = \begin{cases} \frac{z - \sin z}{z^3} & if\ z \neq 0 \\ \frac{1}{6} & if\ z = 0 \end{cases}$$

$\therefore$ The singularity $z = 0$ of the function $f(z)$ is removed ....(3)

From (2) & (3)

$z = 0$ is Removable singularity for $f(z)$

Problem 3:

Prove that $Z = 0$ is Removable Singularity of $f(z) = \frac{e^z - 1}{z}$

Proof:

$$f(z) = \frac{e^z - 1}{z}$$

pat $z = 0$ in (1)

$$(1) \Rightarrow f(z) = \frac{1 - 1}{0}$$

$$= \frac{0}{0}$$

$$f(z) = \infty$$

$\Rightarrow f(z)$ is not analytic at $z = 0$

$\Rightarrow z = 0$ is singularity for $f(z)$.

$$\lim_{z \to a} (z - a) f(z) = \lim_{z \to 0} (z - 0) \frac{e^z - 1}{z}$$

$$\lim_{z \to a} (z - a) f(z) = 0$$

$\therefore z = a$ is a R Removable singularity of $f(z)$

$$f(z) = \frac{e^z - 1}{z} = \frac{1}{2}\left[1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right] - 1$$
$$= \frac{1}{2}\left[\frac{z}{1!} + \frac{z^2}{2!} + \cdots\right]$$
$$= \frac{2}{2}\left[1 + \frac{z}{2!}\right]$$
$$= 1 + \frac{z}{2!}$$

Put $z = 0 \Rightarrow f(0) = 1,$

$z = 0$ is Removable singularity for f(z)

Theorem 2:

Taylor's Theorems for all analytic function

If $f(z)$ is analytic in region Ω containing a. Then it's possible to write $f(z)$ in term of powers of $z - a$ as follows

$$\therefore f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \ldots ..$$
$$\ldots \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n.$$

where $f_n(z)$ is analytic in region Ω.

Proof:

Given, $f(z)$ is analytic in region Ω containing a.

Now consider the function $F(z) = \frac{f(z)-f(a)}{z-a}$ which is analytic in Ω. execpteat $z = a$.

$\Rightarrow F(z)$ is analytic in Ω.

$\Omega' = \Omega - \{a\}$

(i.e.) $z = a$ is singularity of $F(z)$ ………..(1)

(i.e.) $\lim_{z\to a} F(z) = f'(a)$

From the above we choose

$$F(z) = f'(a) \text{ for } z = a$$

$$\therefore \lim_{z\to a}(z-a)F(z) = \lim_{z\to a}(z-a)\frac{f(z)-f(a)}{z-a}$$

$$= f(a) - f(a)$$

$\lim_{z\to a}(z-a)F(z) = 0$ ………….. (3)

From (1) & (3) $z = a$ is Removable

Singularity for the function $F(Z)$

Let $F(z)$ is analytic in Ω'

$\Omega' = \Omega - \{a\}$

and a is Removable singularity of $F(z)$

We know that, (above theorem). $F(z)$ is analytic in a region Ω.

The Necessary & sufficient condition that their exist unique analytic function $F(\text{z})$ in Ω

Such that $F(z) = f(z), \forall z \in \Omega'$ iff $z = a$ is removable singularity of $f(z)$ ………(4)

$\therefore$ their exist unique analytic function $f_1(z)$ in Ω. and $F(z) = f_1(z), \forall z \in \Omega'$

$\therefore$ Consider the function, $f_1(z) = \begin{cases} \frac{f(z)-f(a)}{z-a}, & z \neq a \\ f_1'(a), & z = a \end{cases}$ ……………(5)

(i.e.) $\frac{f(z)-f(a)}{z-a}$ is an analytic in a region Ω expected at $z = a$ $[(ie)z \neq a]$

But $f_1(z) = f_1'(a)$ at $z = a$. Since $f_1'(a)$ is constant.

$\therefore f_1$ is analytic at $z = a$.

$\Rightarrow f_1$ is analytic in a region Ω. from (5).

$$f_1(z)(z-a) = f(z) - f(a)$$
$$\Rightarrow f(z) - f(a) + f_1(z)(z-a)$$

$$f_1(z) = f_1(a) + (z-a)f_2(z)$$

where, $f_2(z) = \begin{cases} \frac{f_1(z)-f_1(a)}{z-a}, & z \neq a \\ f_2'(a), & z = a \end{cases}$

Similarly,

$$f_2(z) = f_2(a) + (z-a)f_3(z)$$
$$\dots \dots \dots \dots$$
$$f_{n-1}(z) = f_{n-1}(a) + (z-a)f_n(z)$$
$$f_n(z) = f_n(a) + (z-a)f_{n+1}(z)$$
$$f(z) = f(a) + (z-a)f_1(z)$$
$$= f(a) + (z-a)[f_1(a) + (z-a)f_2(z)]$$
$$= f(a) + f_1(a)(z-a) + (z-a)^2 f_2(z)$$
$$= f(a) + f_1(a)(z-a) + (z-a)^2[f_2(a) + (z-a)f_3(z)]$$
$$= f(a) + f_1(a)(z-a) + (z-a)^2 f_2(a) + (z-a)^3 f_3(z)$$

Continuing like this we get,

$$f(z) = f(a) + (z-a)f_1(a) + (z-a)^2 f_2(a) + \cdots \dots \dots + (z-a)^{\mathrm{n}} f_{\mathrm{n}}(z) \quad \dots\dots\dots\dots\dots\dots (6)$$

Diff (6) with respect to ' 'z' n times.

$$\therefore f^n(z) = 0 + 0 + \cdots + n!\, f_n(z)$$
$$\Rightarrow f^n(z) = n f_n(z)$$

put $n = 1, f_1(z) = \frac{f'(z)}{1!}$

put $n = 2, f_2(z) = \frac{f''(z)}{2!}$ put $z = a$,

put $n = n, f_n(z) = \frac{f^n(z)}{n!}$

we get,

$$\left.\begin{aligned} f_1(a) &= \frac{f!(a)}{1!} \\ f_2(a) &= \frac{f''(a)}{2!} \\ &\dots\dots\dots\dots\dots. \\ f_n(a) &= \frac{fn(a)}{n!} \end{aligned}\right\} \text{Sub in (6)}$$

(6) $f(z) = f(a) + (z-a)\frac{f!(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots\dots\dots + \frac{f^{n-1}(a)(z-a)^{n-1}}{(n-1)!} + (z-a)^n f_n(z)$

Corollary 1: (Representation of Reminder form)

Express $f_n(z)$ as a line integral.

Proof:

Let c be a circle with center at a contain in a region Ω.

Since $f_n(z)$ is analytic through out c.

∴ We can use Cauchy's integral formula.

$$f_n(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\zeta)d\zeta}{(\zeta - z)} \qquad \dots\dots\dots. (1)$$

By the Taylor's Theorem, we can write, f(z) as.

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots\dots\dots + \frac{f^{n-1}(a)(z-a)^{n-1}}{(n-1)!} + (z-a)^n f_n(z).$$

$$\Rightarrow f_n(z) = \frac{\left\{f(z) - f(a) - (z-a)\frac{f'(a)}{1!} - (z-a)^2\frac{f''(a)}{2!} \dots\dots\dots (z-a)^{n-1}\frac{f^{n-1}(a)}{(n-1)!}\right.}{(z-a)^n}$$

put $z = \zeta$, $f_n(x) = \frac{\left\{f(\zeta) - f(a) - (\zeta-a)\frac{f'(a)}{1!} - (\zeta-a)^2\frac{f''(a)}{2!} \dots\dots\dots (\zeta-a)^{n-1}\frac{f^{n-1}(a)}{(n-1)!}\right.}{(\zeta-a)^n}$ ………….(2)

Sub (2) in (1)

$$f_n(z)=\frac{1}{2\pi i}\int_C \frac{1}{(\zeta-z)}\left[\frac{f(\zeta)}{(\zeta-a)^n}-\frac{f(a)}{(\zeta-a)^n}-\frac{(\zeta-a)f'(a)}{(\zeta-a)^n\,1!}\right.$$
$$\left.-\frac{(\zeta-a)^2f''(a)}{(\zeta-a)^n 2!}\cdots\frac{(\zeta-a)^{n-1}f^{n-1}a]}{(n-1)!\,(\zeta-a)^n}\right]d\zeta$$
$$=\frac{1}{2\pi i}\int_C \frac{1}{(\zeta-i)}\int \frac{f(\zeta)}{(\zeta-a)^n}-\frac{f(a)}{(\zeta-a)^n}-\frac{f'(a)}{(\zeta-a)^{n-1}}-\frac{f''(a)}{(\zeta-a)^{n-2}\,2!}$$
$$-=-\frac{f^{h-1}(a)}{(\mathrm{n}-1)!\,(\zeta-a)}\Bigg]d\zeta$$
$$f_n(z)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)\,d\zeta}{(\zeta-z)(\zeta-a)^n}-\frac{1}{2\pi i}\int_c \frac{f(\mathrm{a})d\zeta}{(\zeta-z)(\zeta-a)^n}-\frac{1}{2\pi i}\int_C \frac{f'(a)d\zeta}{(\zeta-z)(\zeta-a)^{n-1}}\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots.-\frac{1}{2\pi i}\int_C \frac{f^{n-1}(a)}{(\zeta-\mathrm{z})(\zeta-a)(n-1)!}d\zeta \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots(3)$$
$$f_n(z)=\frac{1}{2\pi i}\int \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)^n}-F_n(a)-F_{n-1}(a)\ldots\ldots\ldots\ldots F_1(a) \quad \ldots\ldots\ldots\ldots\ldots.(A)$$

Now, $\frac{1}{(\zeta-z)(\zeta-a)}=\frac{A}{(\zeta-z)}+\frac{B}{(\zeta-a)}$ ....(4)

$1=A(\zeta-a)+B(\zeta-z)$

Put $\zeta=a$,

$$1=A(0)+B(a-z)$$
$$\Rightarrow B=\frac{1}{a-z}$$

put $\zeta=z$

$$1=A(z-a)+B(0)$$
$$\Rightarrow A=\frac{1}{(z-a)},B=\frac{1}{a-z}\ \ldots\ldots\ldots \text{ sub in (4)}$$
$$\Rightarrow \frac{1}{(\zeta-z)(\zeta-a)}=\frac{1}{(z-a)(\zeta-z)}+\frac{1}{(a-z)(\zeta-a)}$$
$$=\frac{1}{(z-a)(\zeta-z)}-\frac{1}{(z-a)(\zeta-a)}$$

$\frac{1}{(\zeta-z)(\zeta-a)} = \frac{1}{(z-a)}\left[\frac{1}{(\zeta-z)} - \frac{1}{(\zeta-a)}\right]$ .... (5)

$$\text{Now, } F_1(a) = \int_C \frac{d\zeta}{(\zeta-z)(\zeta-a)} = \frac{1}{(z-a)}\left[\int_C \frac{d\zeta}{(\zeta-z)} - \int_C \frac{d\zeta}{(\zeta-a)}\right]$$
$$= \frac{1}{(z-a)}[2\pi i - 2\pi i]$$
$$\therefore F_1(a) = \int_C \frac{d\zeta}{(\zeta-z)(\zeta-a)} = 0$$

$F_1(a) = 0 \Rightarrow F_1'(a) = 0$

$We\ know\ that\ , F_n(a) = nF_{n+1}(a)$

put $n = 1$,

$F_1(a) = 1F_2(a)$

$0 = F_2(a) \Rightarrow F_2'(a) = 0$

Put $n = 2$,

$F_2(a) = 2F_3(a)$

$0 = 2F_3(a) \Rightarrow F_3(a) = 0$

$\Rightarrow F_3'(a) = 0$

Sub all in (A)

$$\therefore (A) \Rightarrow f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-a)^n} - 0 \cdots - 0$$

(i.e.) $f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(s)ds}{(s-z)(s-a)^n}$

2.5. Zeros and Poles:

Definition: The zeros of an analytic function:

If a function $f(z) = (z-a)^k f_k(z)$ where k is positive integer. and $f_k(a) \neq 0$.

Then a is said to be zeroes of order (or) multiplies of k.

Note:

1. If $F(z)$ is analytic in the neighbourhood of a point $z = a$.

   then $f(z) = \sum_0^\infty a_n(z-a)^n$. where $a_n = \frac{f^n}{n!}$

$\therefore f(z) = a_0 + (z-a)a_1 + (z-a)^2 a_2 + \cdots$

2. If $a_0 = a_1 = a_2 = \cdots = a_{m-1} = 0, a_m \neq 0$.

Then $f(z)$ is said to have a zero of order m at $z = a$.

3. The zero is said to be simple if $m = 1$.

Zeros Definition:

The zero of an analytic function $f(z)$ is value of z for which $f(z) = 0$.

Definition: Isolated singularity:

If $z = a$ is a singular point of a function $f(z)$ and if their exist a neighbourhood of $z = a$ containing no other singular point of a $f(z)$. Then $z = a$ is said to be an isolated singularity point of $f(z)$.

if how ever $f(z)$ has infinite of singular point in every neighbourhood of $z = a$. Then $z = a$ is non-isolated singular point. But it is a limit point of the set of singular point of f(z).

Definition: Poles

If $\lim_{z\to a} f(z) = \infty$ then $z = a$ is pole of $f(z)$

Poles of an analytic function

An isolated singularity a of $f(z)$ is said to be poles of order k if $f(z) = (z-a)^{-k} f_k(a)$ where $f_k(a) \neq 0$. and $f(z)$ is analytic.

Note:

1. An isolated singularity a of $f(z)$ is called a poles of $f(z)$. If $\lim_{z\to a} f(z) = \infty$ (i.e.,) $f(a) = \infty$.
2. The limit point of zeros of an analytic function is a singular point of the function.
3. If there are n time terms in principle part of $f(z)$ then pole is of order n.
4. pole of order 1 is called simple pole.
5. If $z = a$ is a pole of $f(z)$ then $z = a$ is zero of $\frac{1}{f(z)}$ also the function $g(z) = \frac{1}{f(z)}$. as a removable singularity at $z = a$:

6. Suppose that $f(z)$ is analytic function. and $g(z) = f\left(\frac{1}{z}\right)$

(a) $g(z)$ as a zero at $z = 0$. Then $f(z)$ is a said to have a zero at $z = \infty$.

(b) If $g(z)$ is, has pole at $z = 0$ then $f(z)$ is said to have a pole at $z = \infty$.
(c) If $g(x)$ as a removable singularity at $z = 0$. than $f(z)$ is said to have a removable singularity at z ∞

D) If $\lim_{z\to a} f(z)$ exist finitely then $z = a$ is a removable singularity.

Essential Singularity:

If $\lim_{z\to a} f(z)$ does not exist then $z = a$ is an essential singularity. Note that, a principle part is an infinite series of negative power series.

Essential singularity of a function

Let $f(z)$ be defined in a region Ω consider the condition.

(i) $\lim_{z\to a} |z - a|^{\alpha} |f(z)| = 0, \alpha = \text{real}$.

(ii) $\lim_{z\to a} |z - a^{\beta}|f(z)| = \infty, \beta = \text{real}$

(a) If neither condition (i) nor condition (ii)

holds for any real α then $z = a$ is called essential singularity of f(z).

(b) If condition (i) holds for all α then function $f(z)$ is identically is zero.

(c) There exist an integer h such that condition (i) holds for $\forall\alpha > h$ and condition

(ii) holds $\forall\alpha < h$.

Note:

(1) The limit point of poles of a function is called non-isolated essential singularity of that function

(2) The limit point of zeros of a function is called isolated essential singularity of that function.

(3) The poles of an analytic function are isolated.

(4) The zeros of an analytic function are isolated.

(5) If $f(z)$ has a pole at $z = a$ then $|f(z)| \to \infty$ as $z \to$ a in any manner.

Meromorphic Function:

An analytic function whose only singularities in the finite plane are poles is called a meromorphic function.

Entire Function (or) Integral function:

A function which is analytic everywhere in the finite plane is called an entire function (or) integral function.

Rational function:

If the only singularities of an analytic function including the point at infinity are poles then the function is a rational function.

Note:

1. A function which as no singularity in the finite part of the plane (or) at ∞ is a rational function.
2. The function which is analytic in the whole plane and as a non-essential singularity at ∞ reduce to a polynomial.
3. Any function which is meromorphic in the extended the plane is rational.

Theorem 1:

If $f(z)$ is analytic in region Ω and $f(a)$ to grether with all derivatives $F^{\gamma}(a)$ vanish in Ω. then $f(z) = 0$ in Ω.

Proof:

We know that, Taylor's Theorem,

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \dots \dots .. + \frac{f(n-1(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n \qquad \dots \dots \dots \dots \dots \dots (1)$$

Where $f_n(z)$ is analytic in the region Ω.

By hypothesis, $f(a)$ and all derivatives $f^{\gamma}\,(a)$ vanishes in Ω.

$\therefore (1) \Rightarrow$

$f(z) = (z-a)^n f_n(z)$

$\Rightarrow |f(z)| = |(z-a)^n||f_n(z)| \dots \dots \dots \dots .. (2)$

Let C be a circle, with centre at a and radius γ containing the Ω.

$$\therefore f_n(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - a)^n}, \forall z \in c$$

Taking modulus on both sides, $\Rightarrow |f_n(z)| \leqslant \frac{1}{2\pi}\int_C \frac{|f(\zeta)||d\zeta|}{|(\zeta-z)||(\zeta-a)|^n}$ ....(3)

Since $f(\zeta)$ is andytic in Ω.

$\Rightarrow f(\xi)$ is continuous on c in Ω. Let $M = \max|f(\zeta)|, \forall g \in c$.

$$\therefore |f(\zeta)| \leqslant M, \quad \forall\ \zeta \in C \quad ....(4)$$

$$c: |\zeta - a| = r$$

$$\Rightarrow |\zeta - a|^n = r^n \quad ....(5)$$

Let

$$|\zeta - z| = |\zeta - a + a - z|$$

$$\geq |\zeta - a| - |z - a|$$

$$|\zeta - z| \geq r - |z - a| \quad ....(6)$$

Sub (4) (5) & (6) in (3)

$$\cdot (3) \Rightarrow |f_n(z)| \leqslant \frac{1}{2\pi}\int_c \frac{M|d\zeta|}{\mathrm{r}^n[r - |z - a|]}$$

$$\leqslant \frac{1}{2\pi}\frac{m}{r^n[r-|\ z - ai]}\int_C |d\zeta|$$

$$\leqslant \frac{Mr}{\gamma^n(r-|\ z - a|)}$$

$$\Rightarrow |z - a|^n|f_n(z)| \leq \frac{|z - a|^n mr}{r^n[r - |z - a|]}$$

$$\leqslant \left[\frac{|z - a|}{r}\right]^n \frac{mr}{[r - |z - a|]}$$

$\rightarrow 0$ as $n \rightarrow \infty$ sub in (2).

$\therefore (2) \Rightarrow |f(z)| = 0, \forall z \in C.$

$\Rightarrow f(z) = 0, \forall z =$ inside of c.

$\therefore f(z) = 0$ in Ω $[\therefore c \in \Omega]$

Let $A =$ The set of point of z such that $f(z) = 0$ & $\Omega = A \cup B$

(i.e.) $A = \{z/\ f(z) = 0\}$ & $\Omega = A \cup B$

where $B = \{z/\ \ f(z) \neq 0\}$

To claim: A is an open set.

we know that $A = \{z/f(z) = 0\}$

Let z_0 be any arbitrary point of A.

$\therefore$ *their exist* $r > 0$. Such that, $S_r(z_0) \subset A$.

Since we have already proved $f(z) = 0, \forall z$ inside of c with radius $r > 0$.

$\therefore$ A is contained in some open sphere contained at some aribitary point z_0.

$\therefore$ A is an open set.

To claim:- B is an open set

We know that $B = \{z/f(z) \neq 0\}$

clearly, $f(B) = \{f(z) \neq 0\}$ is an open set.

[$\because$ F is continuous iff $f^{-1}(G)$ is an open set, whenever G is an open].

$\therefore$ B is an open set.

$\therefore$ $\Omega = A \cup B$. Where A & B are disjoint open set.

Since nonempty connected open set is region.

$\Rightarrow$ Ω is connected. ['every region is connected]

Given $f(a) = b$,

$\therefore a \in A\ \ , B \neq \phi$
$\Rightarrow \Omega = A \cup B = A$

$\therefore$ $f(z) = 0$ inside of region Ω.

Theorem 2: (Uniqueness Theorem)

If $f(z)$ and $g(z)$ are analytic in Ω and if $f(z) = g(z)$ on a set which as an assumption accumalation point in Ω. Then $f(z)$ is identically equal to $g(z)$ for all z in Ω.

[(i.e.) $f(z) \equiv g(z), \forall z \in \Omega$]

Proof:

Consider $h(z) = f(z) - g(z)$ is analytic an Ω ....(1)

Since given $f(z)$ & $g(z)$ are analytic in Ω.

From given $f(z) = g(z), \forall z \in S$.

$\therefore h(z) = 0 \ \forall z \in S. \longrightarrow (*)$

$\therefore$ Every z in S is a zeros of f(z).

Let $a \in \Omega$ be a limit point. (accumulation point) of S.

Then the function $h(z)$ can be extended

By a Taylors the theorem about a.

$$\therefore h(z) = h(a) + \frac{h'(a)(z-a)}{1!} + \frac{h''(a)(z-a)^2}{2!} + \cdots \cdots \frac{h^{n-1}(a)(z-a)^{n-1}}{(n-1)!} + h_n(z)(z-a)^n \quad \dots\dots\dots\dots\dots (2)$$

Since $h(z)$ is continuous at $z = a$. and $h(a) = 0$.

Since $z = a$ is a limit point of zeros of $h(z)$.

$\therefore (2) \Rightarrow$

$$h(z) = (z-a)\left[\frac{h'(a)}{1!} + \frac{h''(a)(z-a)}{2!} + \cdots + \frac{h^{n-1}(a)(z-a)^{n-2}}{(n-1)!} + h_n(z)(z-a)^{n-1}\right]$$

$h(z) = (z-a)\,\phi(z)$

where $\phi(z) = \left[\frac{h'(a)}{1!} + \frac{h''(a)(z-a)}{2!} + \cdots + \frac{h^{n-1}(a)(z-a)^{n-2}}{(n-1)!} + h_n(z)(z-a)^{n-1}\right]$ ……(3)

Since Every z in S is zeros of $h(z)$

Every z in S is also a zeros of $\phi(z)$.

$\therefore \phi(z) = 0,\ z \in S.$

$\therefore \phi(z)$ is continuous at $z = a$ and $\phi(a) = 0$ ………(4)

Since $z = a$ is a limit point zeros in $\phi(z)$.

$\therefore$ Put $z = a$ in eqn (3)

we get,

$\phi(\mathrm{a}) = h'(a) + \cdots + hn(a)(a-a)^{n-1}$
$\phi(a) = h'(a) + 0 + 0 + 0 + [(4)]$
$\Rightarrow 0 = h'(a)$ sub in (2). $[\because (4)]$

$\therefore (2) \Rightarrow$

$$h(z) = 0 + 0 + \frac{h''(a)(z-a)^2}{2!} + \cdots + h_n(z)(z-a)^n$$

Repeating this process we get, $h''(a) = 0,\ h'''(a) = 0$ …………..(5)

$\therefore h(z)$ is analytic on Ω and all derivatives

$h^{\gamma}(a) = 0.$ [since (1) & (5)]
$\therefore \Rightarrow h(z) = 0$ in Ω
$\Rightarrow f(z) - g(z) = 0$ in Ω
$\Rightarrow f(z) \equiv g(z)$ in Ω

Theorem 3: (Singular part (or) Principle part of the function)

Express a function $f(z)$ as the sum of two parts of which one is Singular part and the bother one is regular part.

Proof:

Let $z = a$ be a pole of order h for Function $f(z)$.

Then in the neighborhood of $z = a$ we can write, $f(z) = \frac{f_h(z)}{(z-a)^h}$ ....(1)

where $f_h(z)$ is analytic at a. and $f_h(a) \neq 0$.

By Taylors theorem,

$$f_h(z) = B_h + (z-a)\,B_{h-1} + (z-a)^2 B_{h-2} + \cdots \ldots + (z-a)^{\mathrm{h}-1} B_1 + (z-a)^{\mathrm{h}} \phi(z)$$

.... (2)

Where $\phi(\mathrm{z})$ is analytic in the neighbourhal of h.

Sub (2) in (1)

$\therefore$ (1) $\Rightarrow$

$$f(z) = \frac{1}{(z-a)^h}\left[B_h + (z-a)B_{h-1} + (z-a)^2 B_{h-2} + \cdots + (z-a)^{h-1} B_1 + (z-a)^h \phi(z)\right]$$

$$\left[\frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \frac{B_{h-2}}{(z-a)^{h-2}} + - \cdots + \frac{B_1}{(z-a)^1}\right] + \phi(z) \quad \text{........... (3)}$$

The part,

$$\frac{B_h}{(z-a)^h} + \frac{B_{h-1}}{(z-a)^{h-1}} + \frac{B_{h-2}}{(z-a)^{h-2}} + - \cdots + \frac{B_1}{(z-a)^1}$$

is called singular part (or) principle part of $f(z)$ in the neighbourhood of the pole $z = a$ of order h.

[Note:

In The case of an isolated singularity at $z = a$. we can use Lawrence series for $f(z)$

(i.e.) $f(z) = [a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n \cdots + \infty]$

$$+\left[\frac{b_1}{(z-a)}+\frac{b_2}{(z-a)^2}+\cdots+\frac{b_n}{(z-a)^n}+\cdots+\infty\right] \quad 0<r_1<|z-a|]$$

Case :1

Suppose that $b_1 = b_2 = \cdots - b_h = -= 0$.

Then $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^n \cdots + \infty$
we define $F(z) = a_0$ then $z = a$ is called removable singularity of $f(z)$

Case: 2

Suppose that.

$b_1 = b_2 = \cdots = b_{n-1} = 0,\ b_h \neq 0$
$r_1 < |\mathrm{z} - a| < r_2$

Then $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_n(z-a)^{\mathrm{n}} + \cdots + \infty + \left[\frac{bh}{(z-a)^h}\right]$

Then $z = a$ is called a poles of order h.

Hence the singular part continuous only a finite number of terms in $\frac{1}{\mathrm{z}-a}$.

Case: 3

Suppose that, The singular part does not terminate then

$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_b}{(z-a)^{n^1}} + \cdots \infty$ for $r_1 < |z-a| < r_2$

Is an essential singularity of $f(z)$.

Theorem 4: (Weierstrass's Classical Theorem)

An analytic function comes arbitrary closed to any complex value in every neighborhood of an essential singularity.

Proof:

Let $z = a$ be an essential singularity of $f(z)$.

Suppose that the theorem is not true.

(i.e.) their exist complex value $A, \delta > 0$.

Such that, $|f(z) - A| \nless \varepsilon$, for $|z - a| < \delta$.

where, $z = a$ is an essential singularity of $f(z)$.

(i.e.) $|f(z) - A| > \varepsilon$ for $|z - a| < \delta$. ………….. (1)

for any real $\alpha < 0$.

we have

$$\lim_{z\to a}|z-a|^{\alpha}|f(z)-A| \geq \lim_{z\to a}|z-a|^{\alpha}\varepsilon.$$
$$> \frac{\varepsilon}{0} = \infty.$$

(i.e.) $\lim_{z\to a}|z-a|^{\alpha}|f(z)-A| = \infty$

$\therefore z = a$ cannot be an essential singularity of $f(z)$ ……….(2)

For any $\beta > 0$.

$$\lim_{z\to a}|z-a|^{\beta}|f(z)-A| \geq \lim_{z\to a}|z-a|^{\beta}\varepsilon > \varepsilon(0) = 0$$

(i.e.) $\lim_{z\to a}|z-a|^{\beta}|f(z)-A| = 0$ ………..(3)

Hence, $z = a$ cannot be an essential singularity of $f(z) - A$ and so

$$\lim_{z\to a}|z-a||A| = 0 \quad \ldots\ldots\ldots..(4)$$

$$\Rightarrow \lim_{z\to a}|z-a|^{\beta}|f(z)| = \lim_{z\to a}|z-a|^{\beta}|f(z)-A+A|$$

$\leq \lim_{z\to a}|z-a|^{\beta}|f(z)-A| + \lim_{z\to a}|z-a|^{\beta}|A|$

$= 0.0 = 0$

$\lim_{z\to a}|z-a|^{\beta}|f(z)| = 0$ .... (5)

$\therefore z = a$ is not an essential Singularity of $f(z)$

From (2) & (5) this a contradiction to our assumption.

$\therefore |f(z)-A| < \varepsilon$, for $|z-a| < \delta$

Theorem 5:

Show that any function which is meromorphic in the extended plane is rational (or) the quotient of two polynomials.

Proof:

Let $f(z)$ be any meromorphic function

Let, $z_{11}z_2 - z_{1k}$ be poles of order m_1, m_2 ar- $-m_k$ respectively for $f(z)$.

$f(z)$ we can write $f(z) = \frac{p(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}--(z-z_k)^{m_k}}$ ....(1)

where $p(z)$ is analytic $\forall z$.

$\Rightarrow p(z) = f(z)[z-a]^{m_1}[z-a]^{m_2} \ldots [z-a]^{m_k}$ .... (2)

Since $p(z)$ is analytic.
$\therefore$ By Taylor's Theorem,

we can write,

$p(z) = \sum a_n z^n$ .... (3)

but $z = \frac{1}{\zeta}$ in (3)

we get,

$$P\left(\frac{1}{\zeta}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{\zeta^n}$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{\zeta^n}$$

$$\therefore P\left(\frac{1}{\zeta}\right) = a_0 + \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots$$

clearly $z = \infty$ is a pole of $p(z)$.

$\Rightarrow \zeta = 0$ is a pole of $P(1/\zeta)$

$\Rightarrow$ (4) has Finite no. of non-zero terms.

$\Rightarrow$ (3) has finite no. of non-zero terms $\Rightarrow p(z)$ is polynomial in z

$\therefore$ (1) $\Rightarrow f(z) = \frac{\text{poly in } z}{\text{poly in } z}$ = rational function Hence proved.

2.6. The Local Mapping:

Theorem 1:

Let z_j be the zeros of the function $f(z)$ which is analytic in disc Δ. and $f(z)$ does not vanish identically each-zero being counted as many times as its order indicates then for every closed γ curve in Δ which does not passes through a zero.

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

where sum as only a finite no. of terms is not equal to zero.

Proof:

Given that $f(z)$ is analytic function is open circular disc $\triangle$ and $f(z) \neq 0$ in the disc Δ.

Let γ be a closed curve in Δ.

Let us suppose that $f(z)$ has only finite no. of zeors in $\triangle$.

(i.e.)Let $z_1, z_2 \ldots - z_n$ be the finite no. of zero of $f(\mathrm{z})$ inside $\triangle$, each zero being counter according to its degree of multiplicity.

(i.e.) Each zero is repeated has many times has its order indicates.

Since $z_1, z_2 \cdots z_n$ zeros of $f(z)$ and $f(z)$ in analytic.

$\therefore$ we can write,

$f(z) = (z - z_1)(z - z_2) - \cdots (z - z_n) g(z)$

Where $g(z)$ is analytic and not null in $\triangle$.

$\therefore$ Talking log on both sides

$$\therefore \log f(z) = \log(z - z_1) + \log(z - z_2) + \cdots + \log(z - z_n) + \log \mathrm{g}(z) \quad \ldots \ldots (1)$$

Diff (1) with respect to ' z ',

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)} \quad \ldots\ldots\ldots\ldots (2)$$

multiply by $\frac{1}{2\pi i}$ & Integrate on both sides,

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_1} dz + \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_2} dz + \cdots + \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_\mathrm{n}} dz$$

$$+ \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} dz \ldots \ldots \ldots \ldots (3)$$

Hence $g(z)$ is analytic and not null in $\triangle$.

$\Rightarrow g'(z)$ is analytic.

$\Rightarrow \frac{g'(z)}{g(z)}$ is also analytic as

$g(z) \neq 0$

∴ By Cauchy's Theorem.

$$\int_\gamma \frac{g'(z)}{g(z)} dz = 0 \; where \; \frac{g'(z)}{g(z)} \; is \; analytic$$

∴ (3) ⇒

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_1} dz + \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_2} dz + \cdots + \frac{1}{2\pi i}\int_\gamma \frac{1}{z-z_n} dz$$

$$= \sum_j n(\gamma, z_j) \quad \ldots\ldots\ldots(4) \quad \left[\because n(\gamma, a) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}\right]$$

Thus we have proved theorem. If $f(z)$ has a finite no. of zeros. Then eqn (3) is true. if $f(z)$ has infinitely many zeros in Δ

Then for any closed curve D, Δ we can find a smaller disc Δ'

such that $\gamma \leq \Delta' \leq \Delta$. Now, there are only a finite no. of zeros of $f(z)$ in Δ'

Otherwise if there are infinitely many zeros of $f(z)$ in Δ'.

∴ Bolzano Weierstrass's theorem.

They would have a accumulation point in the closure of Δ'. and at this is impossible.

(i.e.) we can find an infinite sequences of zeros.

(i.e.) $\{z_n\}$ be a zeros of $f(z)$.

Such that,

$z_n \to z_0$ (accumlation point) as $n \to \infty$.

(i.e.) $f(z_0) = f[\lim_{n\to\infty}(z_n)]$.

$$= \lim_{n\to\infty} f(z_n)$$
$$= 0$$

Thus the zero z_0 of an analytic function is not isolated.

This is contradiction.

$\therefore$ Δ' contain only a finite no. of zeros apply equation (3) to the disc Δ'

$\therefore$ The zeros outside of the disc Δ ' $n(\gamma, z_j) = 0$.

and Hence not contributed the Sum R.H.S. of equation (3) is written

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, z_j) \quad \ldots\ldots\ldots\ldots(*)$$

Remark:1

(i)The function $\omega = f(z)$ maps γ an to a closed curve Γ in the w plane

$put\ \omega = f(z)$

$$\Rightarrow dw = f'(z)dz$$
$$\Rightarrow f'(z) = \frac{dw}{dz}$$

in above theorem equation$(*)$ $\Rightarrow \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} dz = \sum n(\gamma, z_j)$

(ii) If a & b lies in the same region determined by Γ. Then,

$$\sum n(\gamma, z_j(a)) \neq n(\Gamma, b)$$
$$\therefore n(\gamma, a) = n(\Gamma, b)$$

Thus $f(z)$ takes values a & b equally many times inside γ.

Theorem 2:

Suppose that $f(z)$ is analytic at z_0. $f(z) = w_0$ that $f(z) - w_0$ as a zero of order n at z_0. If $\varepsilon > 0$ is sufficiently small then their exist $\delta > 0$, such that $\forall$ a with $|\mathrm{a} - w_0| < \delta$. The equation $f(z) = a$ as exactly n roots in the disc $|z - z_0| < \varepsilon$.

Proof:

Let us choose $\varepsilon > 0$. Such that $f(z)$ is define and analytic in $|z - z_0| < \varepsilon$

$\therefore$ z_0 is the only zero of $f(z) - \omega_0$ of order n in the neighborhood .... (1)

Let γ be a circle $|z - z_0| < \varepsilon$ and let Γ is image under the mapping $\omega = f(z)$.

(i.e.) Γ is a closed curve.

Since $w_0 \notin \gamma \Rightarrow f(z_0) \notin \Gamma$ or

Hence w_0 is in complements of Γ

$\therefore$ their exist neighborhood,$|w - w_0| < \delta$ and for every ε

$|w - w_0| < \delta$ all the values of a are taken same no. of time inside γ.

But since $f(z) = \omega_0$ as exactly n co-inside in roots inside γ.

$\therefore$ $f(z) = a$ also exactly n roots and every value of a taken n-times.

(i.e.) Every value $a \in |\omega - \omega_0| < \delta$, is taken ' n ' no. of times by $f(\mathrm{z})$ inside γ .

(i.e.) $f(z) = a$ as n roots in the disc $|z - z_0| < \varepsilon$ is understood.

(or) (Another Proof)

Let γ be a circle $|z - z_0| < \varepsilon$ and $f(z)$ is analytic and define for $|z - z_0| \leq \varepsilon$ and given that $f(z_0) = \omega_0$.

$\therefore$ $f(z) - w_0$ as a zeros of a vardar in at $= 0$

Let the image of γ center $f(z)$ be the closed curve Γ in the w plane.

Now $\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz =$ Total no. of zeros of $f(z) - w_0$

$= \text{n}$

Their exist a neighborhoods $|a - w_0| < \delta$ of w_0 contain whole inside Γ.

$$\therefore \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-a}dz = \frac{1}{2\pi i}\int_{\Gamma}\frac{f'(z)dz}{f(z)-w_0}$$

$$\Rightarrow n(\gamma, a) = n(\Gamma, w_0) = n$$

(i.e.) The function takes all values in the neighborhood of the point w_0 equaltly many times inside Γ.

$\therefore f(z) - w_0$ as exactly n roots does every value of a is taken n times inside γ.

Corollary:1

A non-constant analytic function a maps open set onto open sets.

Corollary: 2

If $f(z)$ is analytic at z_0 why with $f'(z_0) \neq 0$ it maps in the neighbourhood of Z_0 conformly and topologically onto region.

2.7. Maximum Principle:

Theorem: 1 (Maximum Modulus principle (or) Maximum principle)

If $f(z)$ is analytic at non-constant in a region Ω. Then its obsolutly value $|f(z)|$ as no maximum in Ω. (or)

If $f(z)$ is non-constant defined and continuous an closed bounded set E and andytic an the interior of E. Then Maximum of $|f(z)|$ on E ss assumed only and theundary of E.

Proof:

Given, E is closed and bounded ⇒ E is compact.

Assume that $|f(z)|$ as a maximum value on E, say at $z_0 \in E$. (interior point of E).

Given $f(z)$ is non-constant and analytic

$\therefore |f(z) - f(z_0)| < \varepsilon$ if $|z - z_0| < \delta$.

(i.e.) $|f(z)| - |f(z_0)|| < |f(z) - f(z_0)| < \varepsilon$

$\Rightarrow |f(z)|$ is continuous.

If z_0 is an interior point. Then $|f(z_0)|$ is the maximum of $|f(z)|$ in $|z - z_0| < \delta$ contain in E.

But it is impossible unless $f(z)$ is a constant in the compliment of the interior of E containing z_0.

The continuity $|f(z)|$ as its maximum on the whole boundary of that compliment and this boundary is non-empty and is contained in the boundary of E.

Thus the maximum of $|f(z)|$ is always obtain only or the boundary of E.

(or) Analytical Proof:

Let c be the boundary of the region, assume that $|f(z)|$ as its Maximum at a point

$z_0 \in \Omega$ [z_0 is an interior of point Ω].

Where z_0 lies inside of c. draw the small circle γ.

$\gamma: |z - z_0| = r$
$|f(z_0)| = M =$ maximum value ……..(1)

∴ from the circle γ we get,

$z - z_0 = re^{i\theta}, 0 \leq \theta \leq 2H$
$\Rightarrow z = z_0 + re^{i\theta}$
$\Rightarrow dz = re^{i\theta} i d\theta$ ………. (2)

By Cauchy's Integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)dz}{z - z_0}$$
$$= \frac{1}{2\pi i}\int_{0}^{2\pi} \frac{f(z_0 + re^{i\theta})}{\gamma e^{i\theta}} \gamma e^{i\theta} \phi d\theta.$$
$$= \frac{1}{2\pi}\int_{0}^{2\pi} f(z_0 + re^{i\theta}) d\theta.$$

$\Rightarrow |f(z_0)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta$ ………… (3)

Since, $|f(z_0)|$ is a maximum value.

$\therefore |f(z_0 + re^{i\theta})| < |f(z_0)|$, for single value of ' θ '.

By continuity of $f(z)$,

$\therefore |f(z_0 + re^{i\theta})| \leq |f(z_0)|$ on a whole finite are

$\Rightarrow$ mean value of $|f(z_0 + re^{io})|$ on δ , $\gamma < |f(z_0)|$

$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta < |f(z_0)|$ ………….. (4)

sub (4) in (3)

$\therefore B \Rightarrow |f(z_0)| < |f(z_0)|$

This is contradiction

$\therefore |f(z)|$ must reduce to a constant and is equal to $|f(z_0)|$ for all sufficiently small circles $|z - z_0| = r$ and hence in the neighbourhood of z_0.

$\therefore$ If follows easily that $f(z)$ must reduce to a constant.

This is $a \Rightarrow <=$ to over assumption

(i.e.) our assumption that $|f(z)|$ has is maximum at an interior point is wrong, Hence $|f(z)|$ is Maximum only on the boundary of Ω.

Theorem 2: (Schwartz Lemma)

If $f(z)$ is analytic $|z| < 1$ and statisfies the condition $|f(z)| \leq 1$ and $f(0) = 0$. Then

(i) $|f(z)| \leq |z|$, for some $z \neq 0$ and $|f'(0)| \leq 1$

(ii) If $|f(z)| = |z|$ (or) If $|f'(z)| = 1$ then $f(z) = cz$.

Where c is a constant whose absolute value is 1 .

Proof:

Let us define the function

$$f_1(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & \text{if } z = 0 \end{cases}$$

$$|f_1(z)| = \left|\frac{f(z)}{z}\right|$$

$$= \frac{|f(z)|}{|z|} \quad [\because f(z) | \leq 1 \text{ on the circle } |z| = \gamma < 1]$$

$$\leq \frac{1}{|z|}$$

$$= \frac{1}{\gamma}$$

(i.e.) $|f_1(z)| \leq \frac{1}{\gamma}$.

By Maximum modulus principle, for $|z| \leq r \Rightarrow |f_1(z)| \leq 1$

$$\Rightarrow \left|\frac{f(z)}{z}\right| \leq 1$$

$$\Rightarrow |f(z)| \leq |z|$$

If $z = 0$.

Then $|f'(0)| \leq 1$

ii) if $|f(z)| = |z|$

$$\Rightarrow \left|\frac{f(z)}{z}\right| = 1$$

(i.e.) $\frac{f(z)}{z} = c$

$\therefore f(z) = cz$ where $|c| = 1$

UNIT –III

THE GENERAL FORM OF CAUCHY'S THEOREM and THE CALCULUS OF RESIDUES: Chains and cycles- Simple Continuity - Homology - The General statement of Cauchy's Theorem - Proof of Cauchy's theorem - Multiply connected regions – The Residue theorem - The argument principle.

Chapter 3: Section 3: 3.1 to 3.8

3. General form of Cauchy's Theorem:

3.1. Chains and cycles

Definition: Chains

An arbitrary formal sum of finite number of finite collection $\gamma_1+\gamma_2+\cdots+\gamma_n$ which need not be an are. Satisty the equations.

$$\int f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \cdots + \int_{\gamma_n} f(z)dz$$

are called chains. A formal sum$a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n$.The arcs $\gamma_1+\gamma_2+\cdots+\gamma_n$ is called chains.

Addition of two chain:

The sum of two chains is design define in the obvious way by just a position it's clear that the additive property of line integral remains valued for arbitrary of sum of two chains.

Identical chains:

Two chains are identical ⇔ they yield (give) the same line integral for all functions $f(z)$.

Set:

The following operations do not alter the identity of chains.

(i) Permutation of two arcs

(ii) Sub division of an arc.

(ill) Fusion of sub arcs into a single are

(iv) Reparametrization of an arc.

(v) collection of opposite arcs

Sub conditions:

(i) For a positive integer n. we write $n\nu = \nu + \nu + \cdots + n$ times

(ii) $-\gamma =$ The - re arc of γ.

(iii) $-n\gamma = -\gamma - \gamma, \dots \dots . . n$ times.

Thus every chain is expressible in the form $a_1\gamma_1, a_2\gamma_2 \dots a_n\gamma_n$ where $a_1, a_2 \dots a_n$ are integers.

The zero chain (or) void chain is a chain in which all $a_i = 0$.

Cycle:

A cycle is a chain which is represented as a sum of closed curve.

Result:

For chains in a region we have.

(i) The integral of an exact differential over a cycle is zero.

(ii) The index of a point with respect to cycle as in the case of a single closed curve.

(iii) If γ_1 and γ_2 are two cycles then $n(\nu_1 + \gamma_2, a) = n(\mu_1, a) + n(\nu_2, a)$

3.2. Simple Connectivity:

Definition:

A region is simply connected if it's complement with respect to the extended complex plane is connected.

Note:

(1) A region is simply connected if it has no poles.

(2) We always take a region in the extended complex plane.

Theorem 1:

A region Ω is simply connected iff $n(\gamma, a) = 0$, for all cycles γ in Ω & for all points ' a ' which does not belong to Ω

Proof:

Necessary Part:

Given Ω is simply connected.

To prove that $n(\gamma, a) = 0. \forall$ cycles. $\gamma \in \Omega$ & $\forall a \notin \Omega$

Given Ω is simply connected

$\Rightarrow \bar{\Omega}$ is connected & contains ∞

$a \notin \Omega \Rightarrow a \in \bar{\Omega}$

$\Rightarrow a \in$ unbounded region determined by γ.

$\Rightarrow n(\gamma, a) = 0, \forall \gamma$ in Ω & $\forall a \notin \Omega$.

Sufficient Part:

Given $n(\gamma, a) = 0$, f cycle γ in Ω & all points a $\notin \Omega$.

$\therefore a \in \bar{\Omega}$

To prove that

The region Ω simply connector For this proof, the complement n is Connected.

Assume that Ω is not simply connected

$\Rightarrow$ complement $\bar{\Omega}$ is not connected

$\therefore$ We can write $\bar{\Omega} = A \cup B$

Where, A & B are non-empty, disjoint closed sets.

Let one of these sets are unbounded. (Since its contains ∞) & the other is bounded

$\therefore$ Let A be the bounded set and $a \in A$

Let $\delta =$ The shortest distance between A and B. $\therefore \delta > 0$.

Cover the whole plane with a net of squares Q of side $< \delta/\sqrt{2}$.

Let us choose the squares such that ' a ' lies at the centre of a square, where $a \in A$.

The boundary of Q is denoted by ∂Q consider, the cycle $\gamma = \sum_j \partial Q_j$

Where the sums ranges over all the squares Q_j in the net which have a point in common with ' A '. Since ' a ' is contained in one & only one of the squares.

$\therefore n(\gamma, a) = 1$

Clearly, γ does not meet B. But if the cancellation are carried out its clear that γ does not meet A. Since, we can omit common sides of squares in γ. Each common side is travelled in opposite direction.

$\therefore \gamma \notin A \cup B = \bar{\Omega}$

$\Rightarrow \gamma \in \Omega$

Also, $a \in A \subset \bar{\Omega} = A \cup B$

$\Rightarrow a \notin \Omega$

Thus, their exist a cycle γ in Ω and a point $a \notin \Omega$ such that $n(\gamma, a) = 1 \neq 0$.

Which is a contradiction to given hypothesis.

$\therefore$ Our assumption is wrong

$\therefore \Omega$ is Simply connected.

Note: If this is a cycle in Ω. Such that $n(\gamma, \mathrm{a}) = 0$. For some a outside of Ω. Then $\frac{1}{z-a}$ is analytic in Ω. Then $\frac{1}{2\pi i}\int \frac{dz}{z-a} = n(\gamma, a) \neq 0$.

3.3. Homology:

Definition:

A cycle γ is an open set Ω is said to be homology to zero with respect to Ω

if $n(\gamma, a) = 0, \forall a \in \bar{\Omega}$. (i.e.,) $a \notin \Omega$

In symbols, we write $\gamma \sim 0$ (mod Ω) (or) $\gamma \sim 0$

Note:

(i) $\gamma_1 \sim \gamma_2 \Leftrightarrow \gamma_1 - \gamma_2 \sim 0$

(ii) Homologous can be add and subtract.

(iii) $\gamma \sim 0(\text{mod}\Omega) \Rightarrow \gamma \sim 0\ (\text{mod}\Omega')\ \Omega' \supset \Omega$

3.4. General Statement of Cauchy's Theorem:

Theorem 1:

If is analytic in Ω, then $\int_\gamma f(z)dz = 0$ for ever cycle γ which is homologous to zero in Ω

In a different formulation, we are claiming that if is such that $\int_\gamma f(z)dz = 0$ holds for a certain collection of analytic functions, namely those of the form 1/(z-a) with a not in Ω, then it holds for all analytic functions in Ω

In combination with theorem 1 (Previous section 3.3) we have the following corollary

Corollary: 1

If $f(z)$ is analytic in a simple connected region Ω then $\int_\gamma f(z)dz = 0$, for all cycle in γ in Ω.

Corollary: 2

If $f(z)$ is analytic and $\neq 0$ in a simply connected region Ω then it is possible to define Single valued analytic branches of $\log f(z)$ and $n\sqrt{f(z)}$ in Ω.

3.5. Proof of Cauchy's Theorem:

Theorem 1:

If $f(z)$ is analytic in Ω. Then $\int_\gamma f(z)dz = 0$. For every cycle γ which is Homologous to zero in Ω.

Proof:

Case: (i)

We assume that Ω is bounded but otherwise arbitrary given $\delta > 0$. we cover the plane of net of square of side δ and we denote by $Q_j,\ j \in J$ be the closed squares in the Ω Since Ω is bounded.

$\therefore J$ is finite for sufficiently small δ. We can make J is a nonempty set.

The union of squares $Q_j, j \in J$

Consists of closed regions oriented boundaries makes the cycle.

[$\sqrt{\delta}$ is a sum of oriented line segments which are sides of exactly one a_j]

$\therefore \sqrt{\delta} = \sum_{j\in J} \partial Q_j$ .... (1)

We denote by Ω the interior of the unions $\bigcup_{j\in J} Q_j$.

Let γ be a cycle which is homdogaus to zero. we choose δ is So small that γ is contained in Ω_δ.

Consider the point $\zeta \in \Omega - \cdots \Omega_\delta$.

(i.e.,) $\zeta \in \Omega$ and $\zeta \notin \Omega_\delta$

It's belongs to at least one Q which is not in Q_j. There is a point $\varepsilon_0 \in Q$ which is not in Ω.

It is possible to joint ζ and ζ by a line segment. Which lies in Q $\therefore$ does not meet Ω_δ.

Since $\gamma$is consider as a point set is contained in Ω_8.

It's follows that $n(\gamma,\zeta) = 0 = n(\gamma,\zeta_0)$

In particular, $n(\gamma,\zeta) = 0, \forall$ points ζ on $\sqrt{\delta}$ ....(2)

Then by Cauchy integral formula for higher derivative

$$\frac{1}{2\pi i}\int_{\partial Q}\frac{f(\zeta)d\zeta}{\zeta - z} = \begin{cases} f(z), f_j = j \\ 0,\ f_j \neq j_0 \end{cases}$$

Z lies also interior of $\sqrt{\delta}$

Then $f(z) = \frac{1}{2\pi i} \int_{\sqrt{\delta}} \frac{f(\zeta) d\zeta}{\zeta - z}$

$$\sqrt{\delta} = \sum_j \partial Q_j$$

$$\int_\gamma f(z) dz = \int_\gamma \frac{1}{2\pi i} \int_{\sqrt{\delta}} \frac{f(\zeta) d\zeta}{\zeta - z} dz$$

$$= \int_{\sqrt{\delta}} \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) dz}{\zeta - z} d\zeta$$

$$= \int_{\sqrt{\delta}} \left[\frac{1}{2\pi i} \int_\gamma \frac{dz}{\zeta - z} \right] f(\zeta) d\zeta$$

$$= \int_{\sqrt{\delta}} \left[\frac{-1}{2\pi i} \int_\delta \frac{dz}{(z - \zeta)} \right] f(\zeta) d\zeta$$

$$= \int_{\sqrt{\delta}} - n(\gamma, \zeta) f(\zeta) d\zeta$$

$$\int_\gamma f(z) dz = 0 \; [\because n(\gamma, \zeta) = 0]$$

for all cycles γ which is homologous to zero, if γ is bounded

Case 2:

If Ω is unbounded, we replaced the Ω' by intersection Ω with disc $|Z| < R$ which is large, enough to contain γ and point ' a ' in the complement of Ω are lies in the disc.

In either case $(n(\gamma, \mathrm{a})) = 0$

So that $\gamma \sim 0 (\mathrm{mod}\ \Omega')$

$\gamma \sim 0\ (\mathrm{mod}\ \Omega)(\Omega' \subset \Omega)$

This theorem is valid for arbitrary Ω, if Ω is unbounded. $\therefore n(\gamma, a) = 0, a \notin \Omega'$.

$\therefore a \in$ outside of the circe ' c '.

3.6. Multiply Connected Regions

Definition: Multi connected Region

A region which is not simply connected is called a multi-connected. (i.e.,) The region which one contains poles then it's called multi-connected region

Finite Connectivity:

A region Ω is said to have finite connectively, n if the complement of Ω has exactly n-complement.

Infinite Connectivity:

The region Ω is said to have the infinite connectivity if the complement has infinitely many components

The Calculus of Residues

3.7. The Residue Theorem:

Definition:

The Residue of $f(x)$ at an isolated singularity ' a ' is a unique complex number R which makes $f(z)-\frac{R}{z-a}$ the derivative of a single valued analytic function in an annulus $0<|z-a|<\delta$

Theorem 1: Cauchy's Residue Theorem

Let $f(z)$ be an analytic except for isolated singularity ' a_j ' in the region Ω.

then, $\frac{1}{2\pi i}\int_\gamma f(z)dz=\sum_j n(\gamma,a_j)\ \underset{z=a_j}{\operatorname{Res}} f(z)$

For any cycle γ which is homologous to zero in Ω and does not pass through any of the point a_j.

Proof:

Case (i):

Let there be only a finite number of isolated

Singularities $a_1,a_2 \dots a_n$ in Ω.

The region Ω obtained by excluding the point ' a_j ' will be denoted by Ω '.

(i.e.,) $\Omega'=\Omega-\{a_1\},\{a_2\}\cdots\{a_n\}$

Let γ be a cycle in Ω which is homologous to zero with respect to Ω

Let c_j be a circle with centre a_j, the radius is >0. Consider the circle,

$$\Gamma=\gamma-\sum_\nu n(\gamma,a_j)c_j$$

$$n(\Gamma,a_k)=n(\gamma,a_k)-n\left[\sum n(\gamma,a_j)c_j,a_k\right]$$

$$=n(\gamma,a_k)-\sum_j n(\gamma,a_j)n(c_j,a_k)$$

$$=n(\gamma,a_k)-n(\gamma,a_k)n(c_k,a_k)$$

$$[n(\because n(c_j,a_k)=0 \text{ for j} \neq k]$$

$$=n(\gamma,a_k)-n(\gamma,a_k)=0.$$

$$\therefore n(\Gamma,a_k)=0$$

Let $a \notin \Omega$.

$\therefore n(\Gamma, a) = n(\gamma, a) - n\left[\sum n(\gamma, a_j)c_j, a\right]$

$= n(\gamma, a) - \sum n(\gamma, a_j)n(c_j, a)]$

$= 0 - 0$ $\quad [\because$ homologous to zero. $\Rightarrow \gamma \sim 0(\text{mod}\Omega) \Rightarrow n(c_j, a) = 0]$

$n(\Gamma, a) = 0 \; \therefore \Gamma$ is a cycle in Ω

Which is homologous to zero (mod Ω) which does not pass through the $a_j^{\prime s}$

By Cauchy's theorem,

$$\int_\Gamma f(z)dz = 0$$

$$\therefore \int_\Gamma f(z)dz = \int_{\gamma - \Sigma n(\gamma, a_j)c_j} f(z)dz = 0$$

$$\Rightarrow \int_\gamma f(z)dz = \int_{\Sigma n(\gamma, a_j)c_j} f(z)dz$$

$$[\because \int_c f(z)dz = \int cf(z)dz]$$

$$= \sum n(\gamma, a_j) \int_{c_j} f(z)dz$$

$$\Rightarrow \frac{1}{2\pi i}\int_\gamma f(z)dz = \sum n(\gamma, a_J)\frac{1}{2\pi i}\int f(z)dz.$$

$$= \sum_j n(\gamma, a_j)\frac{1}{2\pi i}P_jC_j \quad [\because \int_{c_j} f(z)dz = P_j]$$

$$= \sum_j n(\gamma, a_j)R_j$$

$$\left[\text{where } R_j = \frac{1}{2\pi i}\int_{c_j} f(z)dz\right]$$

Case (ii):

There are many infinitely isolated singularities in Ω.

The set of all points ' a ' which $n(\gamma, a) = 0$ is open and contains all points outside of large circle.

The complement is consequently a compact set and hence, it cannot contain more than finite number of isolated singularities a_j's $\therefore n(\gamma, a_j) \neq 0$

Only for a finite number of Singularities, $a_j's$.

∴ Case (i) applies

Definition:

The residue of $f(z)$ at the point ∞ is $R_\infty = \frac{-1}{2\pi i}\int_c f(z)dz$

where, c is simple closed contour enclosed all the finite singular point of $f(z)$.

Theorem :1

If a function $f(z)$ is analytic except at a finite number of singularities including the singularity at ∞. Then prove that the Sum of the Residue of $f(z)$ at the singularities is zero.

Proof:

Let C be a simple closed contour exclosing all the finite number of Singularities of $f(z)$

∴ By Residue's theorom,

$\int_c f(z)dz = 2\pi i[R_1 + R_2 + \cdots + R_n]$ ……….. (1)

we know that $\mathbb{R}_\infty = \frac{-1}{2\pi i}\int_c f(z)dz$ ……….(2)

(1) + (2) ⇒

$$R_1 + R_2 + \cdots + R_n + R_\infty = \frac{1}{2\pi i}\int_C f(z)dz - \frac{1}{2\pi i}\int_c f(z)dz$$
$$= 0$$

3.8. The Argument principle:

Theorem:1

If $f(z)$ is meromorphic function in Ω in the zero's a_j and the poles b_k. Then

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = \sum_j h_j n(\gamma, a_j) - \sum_k P_k n(\gamma, b_k)$$

For every cycle γ which is homologous to zero in Ω and does not passes through any of the zeros (or) poles. (or)

If f is meromorphic in a region Ω with finite number of zeros & poles in Ω What are the singularities of $\frac{f'(z)}{f(z)}$ in Ω and compute the Residues of this function at all the singularities in Ω.

Proof:

Let $z = a_j$ be a zeros of order h_j for meromorphic function $f(z)$ in the region Ω Then in the neighbourhood of a_j. we can write,

$f(z) = (z - a_j)^{h_j} \phi_j(z)$ ………….. (1)

Where $\phi_n(z)$ is analytic $\phi_j(a_j) \neq 0$

∴ Taking log on both sides

∴ (1) ⇒

$\log f(z) = h_j \log(z - a_j) + \log \phi_j(z)$

Diff with respect to z

$$\frac{f'(z)}{f(z)} = h_j \frac{1}{(z - a_j)} + \frac{\phi_j'(z)}{\phi_j(z)}$$

∴ $z = a_j$ is simple pole of $\frac{f'(z)}{f(z)}$ and residues h_j, its true for each a_j of $f(z)$

Let $z = b_k$, be the poles of order P_k for $f(z)$ in the region Ω. Then in the neighbourhood of b_k, we can write

$f(z) = \frac{p_k}{(z-b_k)^{p_k}} g_k(z)$ ………….. (2)

where, $g_k(z)$ is analytic in Ω and $g_k(z) \neq 0$.

∴ Taking log on both sides

(2) ⇒ $\log f(z) = -P_k \log(z - b_k) + \log g_k(z)$

Diff w.r.to ' z '

$$\frac{f'(z)}{f(z)} = \frac{-p_k}{(z - b_k)} + \frac{g'_k(z)}{g_k(z)}$$

∴ $z = b_k$ is a Simple poe of $\frac{f'(z)}{f(z)}$ with Residues - P_k, it is true for each b_k of $f(z)$.

∴ Applying the Residues theorem, we get,

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_{j=1}^{n} h_j n(\gamma, a_j) - \sum_{k=1}^{n} P_k n(\gamma, b_k)$$

Where, each zeros and poles are counted according to meets degree of multiplicity.

Corollary: 1

If $f(z)$ is analytic within & on a simple closed contour c and $f(z)$ is non zero on c Then, $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$

Where, N is a number of zeros of $f(z)$ inside c, and p is a number of poles of f(z) inside C.

Proof:

By previous theorem

$\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}dz = \sum\ n(c, a_j)l_j - \sum\ n(c, b_k)m_k$ .... (1)

We know that,

from given where c is a simple closed curve and also a_j and b_k lies inside c.

$\therefore n(c, a_j) = 1$ and n $(c, b_k) = 1$

$\therefore$ Sub in (1), we get,

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = \sum\ l_j - \sum\ m_k$$

$$= (l_1 + l_2 + \cdots) - (m_1 + m_2 + \cdots)$$

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N - p \rightarrow (*)$$

Corollary: 2

If $f(z)$ is analytic with a simple closed contour c except at a finite number of poles inside cx if $f(z)$ is non-zerb on c. Then,

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N = \text{ Number of zero's of } f(z) \text{ inside ' } c \text{ '.}$$

Proof:

Since $f(z)$ is analytic & it has no Singularities (or) poles.

$\therefore P =$ number of poles $= 0$

$\therefore$ By (corollary 1) (*)

$$\Rightarrow \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N - 0$$

$$= \text{ Number of zeros of } f(z)\ inside\ C$$

Theorem: 2

The another proof of the Argument principle

If (i) $f(z)$ is analytic on a simple closed contour C.

(ii) $f(z)$ is meromorphic inside c

(iii) $f($ (z) has no zeros on c.

Then,

$$N - P = \frac{1}{2\pi}\Delta_c \arg f(z)$$

Where, $N =$ number of zeros of $f(z)$ inside c

$p =$ number of poles of $f(z)$ inside c.

Δ_c arg $f(z) =$ The change in argument of $f(z)$ as az describes c.

Proof:

By corollary

$$N - p = \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz$$

$$= \frac{1}{2\pi i}[\log(z)]_c$$

$$= \frac{1}{2\pi i}\Delta_c[\log f(z)]$$

$$= \frac{1}{2\pi i}\Delta_c[\log|f(z)| + i\arg f(z)]$$

$$= \frac{1}{2\pi i}[0 + i\,\Delta_c \arg f(z)]$$

$$= \frac{1}{2\pi i}[i\Delta_c \arg f(z)]$$

$$N - P = \frac{1}{2\pi}[\Delta_c \arg f(z)]$$

Theorem: 3 (Rouche's Theorem)

Let γ be homologous to zero in Ω and Such that $n(\gamma, z)$ is either zero or one for any point z not on γ. Suppose that $f(z)$ *and* $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zero's enclosed by γ .

Proof:

We know that,

(i)If $f(z)$ *and* $g(z)$ are analytic within & on a Simple contour γ.

(ii) $|f(z)| > |g(z)|$ on γ

Thun $f(z), f(z) + g(z)$ have the Some number of zeros inside %.

Step 1:

To prove that $f(z)$ has no zeros on γ . If possible for some point ' a ' no γ

$f(a) = 0.$

$\therefore By(ii) \Rightarrow |f(z)| > \lg(z)|$

$\Rightarrow |f(a)| > |g(a)|$

$\Rightarrow 0 > |g(a)|$

$\Rightarrow |g(a)| < 0$

This is a contradiction. $\therefore f(z)$ has no zeros on γ

Step: 2

To prove that, $f(z) + g(z)$ has no zeros onγ. Suppose that $f(z) + g(z)$ has a zero ' a ' on γ.

$\therefore f(a) + g(a) = 0$

$\Rightarrow f(a) = -g(a)$

$\Rightarrow |f(a)| = |-g(a)|$

$\Rightarrow |f(a)| = |g(a)|$

This is a contradiction $[\because |f(z)| >| g(z)]]$ $\therefore f(z) + g(z)$ has no zeros on γ.

Step: 3

Let N_1 & N_2 be the number of zero's of $f(z) + g(z)$ *and* $f(z)$ inside y respectively. Since $f(z), g(z), f(z) + g(z)$ are analytic within & on γ.

$\therefore$ Number of poles of $f(z) + g(z)$ inside $\gamma = 0$ and also, Number of poles of $f(z)$ inside $\gamma = 0$. $\therefore$ The argument principle

$$N_1 - 0 = \frac{1}{2\pi} \Delta_\gamma \arg(f(z) + g(z))$$

$$8N_2 - 0 = \frac{1}{2\pi} \Delta_\gamma \arg f(z).$$

$$N_1 - N_2 = \frac{1}{2\pi} \left[\Delta_\gamma \arg(f(z) + g(z)) - \Delta_\gamma \arg f(z)\right]$$

$$= \frac{1}{2\pi} \left[\Delta_\gamma \arg[f[1 + g/f]] - \Delta_\gamma \arg f\right]$$

$$= \frac{1}{2\pi} \left[\Delta_\gamma \arg f + \Delta_\gamma \arg[1 + g/f] - \Delta_\gamma \arg f\right]$$

Step: 4

To prove that $\frac{1}{2\pi}\Delta_\gamma \arg[1+g/f]=0.$

$$\begin{aligned} &\text{Let } w = 1+g/f \\ &w-1 = g/f \Rightarrow |w-1| = |g/f| \\ &\qquad\qquad\quad \Rightarrow |w-1| < 1 \end{aligned}$$

$\therefore\ \omega = 1+g/f$ lies inside the circle with center at 1 and radius 1.

$\therefore\ \arg\omega = \arg(1+g/f)$

Returns to the same value after z describes γ.

$\therefore\ \Delta_\gamma \arg\omega = 0 \Rightarrow \Delta_\gamma \arg[1+g/f]=0$

Sub in (1), $\Rightarrow N_1 - N_2 = 0 \Rightarrow N_1 = N_2$

$$\Rightarrow \left.\begin{matrix}\text{Number of zeros of} \\ f(z)+g(z) \text{ inside } \gamma\end{matrix}\right\} = \begin{matrix}\text{number of zeros} \\ \text{of } f(z) \text{ inside } \gamma\end{matrix}$$

Step: 5

proof of Main Theorem:

Give $|f(z)-g(z)| < |f(z)|,$

$$\begin{aligned} &\Rightarrow |g(z)-f(z)| < |f(z)| \\ &\quad |f(z)| > |g(z)-f(z)| \text{ on } \gamma \end{aligned}$$

where $G(z) = g(z)-f(z)$

$\therefore\ By$ (A)

$$\Rightarrow \left.\begin{matrix}\text{Number of zeros of} \\ f(z) \text{ inside } \gamma\end{matrix}\right\} = \begin{matrix}\text{number of zeros} \\ \text{of } f(z)+g(z) \text{inside } \gamma\end{matrix}$$

=Number of zeros of $f(z)+g(z)-f(z)$ inside γ

= Number of zero's of $g(z)$ inside γ.

Residues:

Theorem 1:

If $\int_{c_j}\left(f(z)-\frac{-R_j}{z-a_j}\right)dz$ is of zero period then the constant R_j which is the co-eff of $\frac{1}{z-a_j}$ is called the residue of $f(z)$ at $z=a_j$.

Proof:

Let $f(z)$ be an analytic function in Ω except for a finite no. of singularities $a_1, a_2 \ldots a_n$.

Let Ω' be the region obtained by excluding the pts. a_j

(i.e.,) $\Omega' = \Omega - \{a_1\} \cdots \{a_n\}$.

Let c_j be a circle about a_j of radius δ_j Let $P_j = \int_{c_j} f(z)dz$. which is the period of $f(z)$

Let $f(z)$ be a particular function with a period $2\pi i$.

(i.e.,) $\int_{c_j}\frac{dz}{z-a_j} = 2\pi i$, and $c_j: |z-a_j| = \delta_j$

Let $R_j = \frac{P_j}{2\pi i}$,

$$\int_{c_j}\left(f(z)-\frac{R_j}{z-a_j}\right)dz = \int_{c_j} f(z)dz - R_j\int_{c_j}\frac{dz}{z-a_j}$$
$$= P_j - \frac{P_j}{2\pi \mathrm{i}} \times 2\pi \mathrm{i}$$
$$= P_j - P_j$$
$$= 0$$

Theorem 2:

If $f(z)$ is analytic in $\Omega' = \Omega - \{a\}$ where a is an isolated singularity then their exist a unique complex no. R such that $f(z) - \frac{R}{z-a}$ in the derivative of an analytic function in Ω.

Proof:

Since $f(z)$ has an isolated singularity at $z = a$

∴ we can find a b>0.such that $f(x)$ is analytic in the annulus $0 < 1z - a \, |< \delta$

Let c be the circle $c\colon |z - a| = \text{l}$, where $1 < \delta$

Write $R = \frac{1}{2\pi \text{i}} \int_C f(z)dz$ …………..(1)

Now, we consider,

$$\int_c \left(f(z) - \frac{R}{z-a}\right) dz = \int_c f(z)dz - \int_c \frac{pdz}{z-a}$$

$$= \int_c f(z)dz - 2\pi i p \quad \because \int_c \frac{dz}{z-a} = 2\pi i$$

$$= \int_c f(z)dz - \int_c f(z)dz$$

$$= 0.$$

$\Rightarrow f(z) - \frac{R}{z-a}$ is derivative of an analytic function in the annulus $0 < |z - a| < \delta$.

To prove that: The uniqueness of R

If sufficives to show that $\int_{c_1} f(z)dz = \int_{c_2} f(\text{z})dz$

where c_1, c_2 are two circles $0 < |z - a| < l_i$, $i = 1,2 \ldots$

Since $f(z)$ is analytic in $0 < |z - a| < \delta$. $f(z)$ is analytic in every closed curve Γ in this region.

(i.e.,) by Cauchy's Theorem,

$$\int_\Gamma f(z)dz = 0 \quad \ldots \ldots \ldots (2)$$

choosing, points P, q on c_1, c_2

Consider the closet curve Γ composed of c, (from p to p in the anticloctwiso sense) and the straight line $\overline{qp}$.

$$\int_{\Gamma} f(z)dz = \int_{c_1} f(z)dz + \int_{pq} f(z)dz + \int_{-c_2} f(z)dz$$

$$0 = \int_{c_1} f(z)dz - \int_{c_2} f(z)dz + \int_{qp} f(z)dz$$

$$\int_{C_1} f(z)dz = \int_{c_2} f(z)dz$$

(i.e.,) $\int_C f(z)dz$ is independent of the radius of the circle.

$\therefore$ R is a uniquely determined complex number.

Results:

Type-I Poles of $f(z) = \frac{P(z)}{Q(z)}$ are gen. by $Q(z) = 0$

(i.e.,) $\Pr = 0$.

Type-II: Res $f(z) =$ coefficient of $\frac{1}{z-a}$ in the $z = a$ of $f(z)$ is

Laurent expression of $f(z)$ is power of $(z - a)$

Method of Finding:

The following are the various method of calculation of residue under situation.

Method: 1

$\underset{z=a}{\text{Res}} f(z)$ can be calculated directed by evaluating $\frac{1}{2\pi i}\int_C f(z)dz$.

Choosing a suitable small circle c around $z = a$.

Examples:

1. Find the Residue of $f(z) = \frac{2z+1}{z^2-z-2}$.

Solution:

$$f(z) = \frac{2z+1}{z^2-z-2}$$

poles of $f(z)$:

$$z^2 - z - 2 = 0$$
$$(z+1)(z-2) = 0$$

$\therefore z = -1,2$ are poles.

$$\frac{A}{z+1} + \frac{B}{z-2} = \frac{2z+1}{(z+1)(z-2)}$$
$$A(z-2) + B(z+1) = 2z+1$$
put $z = 2$,
$$\Rightarrow 5 = 3B \Rightarrow B = \frac{5}{3}, \text{ Put } z = -1$$
$$\therefore f(z) = \frac{1/3}{z+1} + \frac{5/3}{z-2}$$

Res. of $f(z)$:

at $z = -1$,

$$\underset{z=-1}{\text{Res}} f(z) = \frac{1}{2\pi i}\int_{c=-1} f(z)dz$$
$$= \frac{1}{2\pi i}\int_{c_1}\left(\frac{1}{3(z+1)} + \frac{5}{3(z-2)}\right)dz$$

where c_1 in the circle centre -1 and radius 1 .

$$= \frac{1}{2\pi i}\left[\frac{1}{3}\int_{c_1}\frac{1}{(z+1)}dz + \frac{5}{3}\int_{1} : |z-(-1)| = 1\right.$$

$$= \frac{1}{2\pi i}\left[\frac{1}{3}2\pi in(c_1-1) + \frac{5}{3}2\pi in(c_1, 2)\right]$$

$\therefore \operatorname{Res}_{z=-1} f(z) = 1/3.$

Similarly, $\operatorname{Res}_{z=2} f(z) = 5/3$

$z = 2$

2. Find the poles of following function and Residue at this poles $\frac{z+1}{z^2-2z}$.

Solution:

$$f(z) = \frac{z+1}{z^2-2z}$$

poles of $f(z)$:

$z^2 - 2z = 0$

$\Rightarrow z(z-2) = 0$

$\Rightarrow z = 0,2$ are poles.

[Let $\frac{z+1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2}$

$\Rightarrow A(z-2) + B(z) = z + 1$

put $Z = 2,\ 2B = 3 \Rightarrow B = 3/2$

P ut $z = 0, -2A = 1 \Rightarrow A = -1/2.$

$\therefore f(z) = \frac{-1/2}{z} + \frac{3/2}{(z-2)}$ sub in (0)]

Res. of $f(z)$ at $z = 0$, Res $f(z) = \frac{1}{2\pi i}\int_{c_1} f(z)dz$

$= \frac{1}{2\pi i} \int_{c_1} \frac{z+1}{z(z-2)} dz$ .... (1)

$$\underset{z=0}{\text{Res}} f(z) = \frac{1}{2\pi i} \int_{c_1} \left(\frac{-1}{2z} + \frac{3}{2(z-2)} \right) dz,$$

$$= \frac{1}{2\pi i} \int_{c_1 i|z-d=1}^{2z} \frac{-1}{2z} dz + \frac{1}{2\pi i} \int_{c_1} \frac{3}{2(z-2)} dz$$

$$= \frac{1}{2\pi i} \left[\frac{-1}{2} \int_{c_1} \frac{1}{z} dz + \frac{3}{2} \int_1 \frac{1}{z-2} dz \right]$$

$$= \frac{1}{2\pi i} \left[-\frac{1}{2} 2\pi i n(c_1, 0) + \frac{3}{2} 2\pi i n(c_1, 2) \right]$$

$$\underset{z=0}{\text{Res}} f(z) = -\frac{1}{2}$$

Similarly, Res $f(z) = 3/2$.

Model - II:

If $z - a$ is a simple pole of $f(z)$.

then,

(a)$\underset{z=a}{\text{Res}}\ f(z) = \lim_{z \to a} (z-a) f(z)$

(b) $\underset{z=a}{\text{Res}}\ f(z) = \lim_{z \to a} \frac{P(z)}{Q'(z)} = \frac{P(a)}{Q'(a)}$

Problem 1:

1. Find the residue of $f(z) = \frac{1}{z^2+5z+6}$.

Solution:

Let $f(z) = \frac{1}{z^2+5z+6}$

poles of $f(z)$:

$$z^2 + 5z + 6 = 0$$
$$(z+3)(z+2) = 0$$

$\therefore$ poles are $z = -2, -3$

Res. of $f(z)$ at $z = -2$

$$\begin{aligned} \text{Res } f(z) &= \lim_{z \to a} (z-a) f(z) \\ &= \lim_{z \to -2} (z+2) \cdot \frac{1}{(z+3)(z+2)} \\ &= \lim_{z \to -2} \frac{1}{(z+3)} \\ &= -1 \end{aligned}$$

at $z = -3$,

$$\begin{aligned} \text{Res } f(z) &= \lim_{z \to -3} (z+3) \frac{1}{(z+3)(z+2)} \\ &= \lim_{z \to -3} \frac{1}{(z+2)} \\ &= -1 \end{aligned}$$

Problem 2:

$f(z) = \frac{1-e^{2z}}{z^4}$.

Solution:

$$\begin{aligned} f(z) &= \frac{1-e^{2z}}{z^4} \\ &= \frac{1 - \left[1 + \frac{2z}{1!} + \frac{(2z)^2}{2!} + \cdots\right]}{z^4} \end{aligned}$$

$$\begin{aligned} f(z) &= -\left[\frac{2z}{z^4} + \frac{4z^2}{2z^4} + \frac{8z^3}{6z^4} + \cdots\right] \\ &= -\left[\frac{2}{z^3} + \frac{2}{z^2} + \frac{4}{3z} + \frac{2}{3} + \cdots\right] \end{aligned}$$

Res. of. $f(z)$ at $z = 0$.

$\underset{z=0}{\text{Res}}\ f(z) =$ co-eff. of $\frac{1}{z-0}$ in $f(z)$

$= -4/3$

Problem 3:

$f(z) = \tanh z$

Solution:

$$f(z) = \tanh z = \frac{\sinh z}{\cosh(z)} = \frac{p(z)}{Q(z)}$$

poles of $f(z)$:

$$\begin{aligned} \cosh hz &= 0. \\ \Rightarrow \cosh hz &= \cos(2n+1)\pi/2 \\ \cos iz &= \cos(2n+1)\pi/2 \end{aligned} \quad \begin{matrix} [\cos ix = \cosh x \\ \sin ix = i\sinh x] \end{matrix}$$

$iz = (2n+1)\pi/2$

Pole $z = \frac{1}{i}(2n+1)\pi/2$

$iz = (2n+1)\pi/2$

pole: $z = \frac{1}{i}(2n+1)\pi/2$

$$\frac{P(z)}{Q'(z)} = \frac{\sinh z}{\sinh z} = 1\ [d(\cosh z) - \sinh z]$$

Res of $f(z)$: $\left[\underset{z\,=\,a}{\text{Res}}(z) = \lim_{z\to a} \frac{P(z)}{Q'(z)}\right]$

at $z = \frac{1}{i}(2n+1)\pi/2$

Res $f(z) =$ pes $\tan hz = \lim 1 = 1$

$z = -i(2n+1)\pi/2 \quad z = -i(2n+1)\pi/2 \quad z \to -i(2n+1)\pi/2$

Problem 4:

$$f(z) = \frac{1}{\sin^2 z}$$

Solution:

$$f(z) = \frac{1}{\sin^2 z} = \frac{1}{\frac{1}{2}(1-\cos 2z)} = \frac{2}{1-\cos 2z} = \frac{P(z)}{2(z)}$$

poles of $f(z)$:

$$1 - \cos 2z = 0$$
$$\cos 2z = 1$$
$$\cos 2z = \cos 2n\pi$$
$$2z = 2\mathrm{n}\pi$$
$$z = n\pi,\ n = 0, \pm 1, \pm 2 \ldots$$

are simple pole

$$\frac{P(z)}{Q'(z)} = \frac{2}{2\sin 2z} = \frac{1}{\sin 2z}$$

Res of $f(z)$:

$$\text{Res}\, f(z) = \lim_{z\to a} \frac{P(z)}{Q'(z)}$$
$$\therefore \text{Res}\, f(z) = \lim_{z\to n\pi} \frac{1}{\sin Bz}$$
$$z = n\pi$$
$$= \frac{1}{\infty}$$
$$= 0$$

Exercises:

1.Evaluate $\int_C \frac{2z^2+z}{z^2-1} dz$

Where (i) c in the circle $|z-1|=1$

(ii) C In the circle $|z|=2$

2.$f(z)=\cot z=\dfrac{\cos z}{\sin z}$

3. $f(z)=\tan z=\dfrac{\sin z}{\cos z}$

4. $f(z)=\frac{z+1}{z^2+9}$

5.obtain the residue of $e^z((z-a)(z-b)$ at its poles in both the cases $a\neq b \& a=b$

6. compute $\int_0^\pi \log\sin\theta\, d\theta$ using residue calculus.

Model-III

If $z=a$ is a pole of order m for $f(z)$ then,

Res $f(z)=\lim_{z\to a}\frac{1}{(m-1)!}D^{m-1}[(z-a)^m f(z)]$

Problem 1:

$$f(z)=\frac{e^{2z}}{(z-1)^2}$$

Solution:

$$f(z)=\frac{e^{2z}}{(z-1)^2}$$

poles of $f(z)$:

$(z-1)^2=0\Rightarrow z=1,1$

$z=1$ in a pole of order 2.

Res of $f(z)$:

$$z = 1, \ \operatorname*{Res}_{z=0} f(z) = \lim_{z \to a} \frac{1}{(m-1)!} D^{m-1}[(z-a)^m f(z)]$$

$$= \lim_{z \to 1} \frac{1}{1!} D' \left[(z-1)^2 \frac{e^{2z}}{(z-1)^2} \right] \ where\ m = 2$$

$$= \lim_{z \to 1} e^{2z} \cdot 2$$

$$= 2e^2$$

Problem 2:

$$f(z) = \frac{1}{2^m (1-z)^n}$$

Solution:

$$f(z) = \frac{1}{2^m (1-z)^n}$$

Poles of $f(z)$:

$$z^m (1-z)^n = 0$$

$$z^m = 0, (1-z)^n = 0$$

$z = 0$ is a pole of order m.

$z = 1$ is a pole of order n

Res of $f(z)$:

at $z = 0$.

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z \to 0} \frac{1}{(m-1)!} D^{m-1} \left[(z-0)^m \frac{1}{z^m (1-z)^n} \right]$$

$$= \lim_{z\to 0} \frac{1}{(m-1)!} D^{m-1} \frac{1}{(1-z)^n}$$

$$= \frac{1}{(m-1)!} \lim_{z\to 0} \left(-1\right)^n n(n+1) \cdots (m-1) \text{ times } (z-1)^{-n(m-1)} \, (-1)^n (-1)^{m-1}$$

$$= \frac{1}{(m-1)!} (n+1-1)(n+2-1) \cdots (n+m-1-1)$$

$$= \frac{(-1)^n 1}{(m-1)!} n(n+1) \cdots (n+m-2).$$

at $z = 1$,

$$\operatorname*{Res}_{z=1} \; f(z) = \lim_{z\to 1} \frac{1}{(n-1)!} D^{n-1} (z-1)^n \frac{1}{z^m (1-z)^n}$$

$$= \frac{1}{(n-1)!} \lim_{z\to 1} (-1)^n D^{n-1} z^{-m} (-1)^n$$

$$= \frac{1}{(n-1)!} \lim_{z\to 1} (-1)^n (-m)(-m-1) \ldots (n-1) \text{ times } z^{-m-n+1}$$

$$= \frac{1}{(n-1)!} (-1^n (-1)^{m-1} m(m+1) \cdots (m+n-2).$$

When is the differential $f(z)dz$ exact in a region. (or)

Prove that $\int_\gamma f(z)dz$ with continuous of depends only on the end points of the γ .

f is the derivative of an analytic function in Ω.

Proof:

$$f(z)dz = f(z)[dx + idy]$$
$$= f(z)dx + if(z)dy$$

$\int_\gamma f(z)dz$ depends only on the end points of γ

A function F(z) in Ω such that

$$\frac{\partial}{\partial x}F(z) = f(z) \text{ and } \frac{\partial F(z)}{\partial y} = if(z) \qquad [\because \text{Theorem}(1)]$$

$$\text{(i.e.,) } \frac{1}{i}\frac{\partial}{\partial y}F(z) = f(z)$$

$$\text{(i.e.,) } -i\frac{\partial}{\partial y}F(z) = f(z).$$

$$\therefore \frac{\partial F(z)}{\partial x} = -i\frac{\partial}{\theta y}F(z)(zf(z))$$

Thus (i) $F(z)$ Satisfies cauchy Riemann eqns.

(ii) Given that $f(z)$ is continuous

$\Rightarrow \frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous in Ω.

From (i) and (ii) $F(z)$ is analytic function $F(z)$

Problem 3:

Compute $\int_\gamma xdz$ where γ is the directed line segment from 0 to $1 + i$.

Proof:

Equation of γ is $y < x$

$$\left[\because \frac{y-0}{1-0} = \frac{x-0}{1-0}\right]$$

$$\therefore dy = dx.$$

$$\int_\gamma x\,dz = \int_\gamma x\,(dx + idy)$$

$$= \int_\gamma (dx + idx)$$

$$= \int_\gamma x\,(1 + i)$$

$$= (1+i)\int_{x=0}^{1} x\,dx$$

$$= (1+i)\left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}(1+i)$$

Problem 4:

Compute $\int_{|z|=2} z^n\,(1-z)^m dz$, where m and n are positive integers.

Proof:

Here γ :|z|=2 is a simple closed curve f(z)= $z^n\,(1-z)^m$ is analytic everywhere and hence analytic inside in all γ

By Cauchy's Fundamental theorem,

$$\int_{\gamma} f(z)\,dz = 0$$

$$\int_{|z|=2} z^n\,(1-z)^m dz$$

Note:

If every cycle γ belong to Ω in the line combination of the cycle $\gamma_1, \gamma_2, \ldots\ldots.\gamma_{n-1}$

$\gamma = c_1\gamma_1 + c_2\,\gamma_2 + \ldots\ldots + c_{n-1}\gamma_{n-1}$

We obtain for any analytic function in Ω

$$\int_{\gamma} f(z)\,dz = \int_{c_1\gamma_1+c_2\,\gamma_2+\,\ldots\ldots+c_{n-1}\gamma_{n-1}} f(z)\,dz$$

$=\int_{c_1\gamma_1} f(z)dz + \int_{c_2\,\gamma_2} f(z)dz + \cdots \ldots \ldots . \int_{c_{n-1}\gamma_{n-1}} f(z)dz$

$=c_1 \int_{\gamma_1} f(z)dz + c_2 \int_{\gamma_2} f(z)dz + \cdots \ldots \ldots . +c_{n-1} \int_{\gamma_{n-1}} f(z)dz$

$= c_1p_1+c_2p_2+\ldots\ldots\ldots+c_{n-1}p_{n-1}$

Where $p_i = \int_{\gamma_i} f(z)\, dz$ i=1 to n-1

This integrals depend only on the function and not on γ

They are called the modulus of periodicity and differential of the f dz the period of indefinite integral.

UNIT-IV:

Evaluation of Definite Integrals and Harmonic Functions: Evaluation of definite integrals - Definition of Harmonic function and basic properties – The Mean value property - Poisson formula.

Chapter 4: Section 4: 4.1 to 4.4

4. Evaluation of Definite Integrals and Harmonic Function:

4.1. Evaluation of Definite Integrals:

Type: 1

Evaluate $\int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta$ where $R(\sin\theta, \cos\theta)$ is a rational function of $\sin\theta$ and $\cos\theta$.

(or)

Explain a general method of Evaluating $\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta$. where R is a rational function of two real variables.

Proof:

Put $z = e^{i\theta}$ (or) $|z| = 1,\ 0 \leq \theta \leq 2\pi$

$$\Rightarrow dz = e^{i\theta} i d\theta.,d\theta = \frac{dz}{e^{i\theta} i}$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

Let $z = e^{i\theta}$

$$= \cos\theta + i\sin\theta$$

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

$\therefore \cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$ and $\sin\theta = \frac{1}{2i}\left[z - \frac{1}{z}\right]$

Contour in $c_i|z| = 1$.

$$\therefore \int_0^{2\pi} R(\sin\theta, \cos\theta)d\theta = \int_c R\left[\frac{1}{2i}\left[z - \frac{1}{z}\right], \frac{1}{2}\left[z + \frac{1}{2}\right]\right]\frac{dz}{iz}$$

$$= 2\pi i[R_1 + R_2 + \cdots - \cdot]$$

Problems:

1.Evaluate $\int_0^{\pi} \frac{d\theta}{a+\cos\theta}, a > 1$

Solution:

Let $I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a\cos\theta}$ ………(1)

To find I : $[\because f(2\pi-\theta) = f(\theta)]$

Let $c: |z| = 1$

put $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$$dz = e^{i\theta} i d\theta$$

$$d\theta = \frac{dz}{iz}$$

and also,

$$\cos\theta = \frac{1}{2}\left[z + \frac{1}{2}\right]$$

$$\therefore I = \frac{1}{2}\int_0^{2\pi} \frac{1}{a + \frac{1}{2}\left[z+\frac{1}{2}\right]} \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_c \frac{1}{a + \frac{1}{2}\left[\frac{z^2+1}{z}\right]} \frac{dz}{z}$$

$$= \frac{1}{2i}\int_c \frac{1}{\frac{2az + z^2 + 1}{2z}} \frac{dz}{z}$$

$$= \frac{1}{2i}\int_c \frac{2z}{2az + z^2 + 1} \cdot \frac{dz}{z}$$

$$= \frac{2}{2i}\int_C \frac{dz}{2az + z^2 + 1}$$

$$= \frac{1}{i}[2\pi i(R_1 + R_2 + \cdots -)]$$

$$=[2\pi(R_1 + R_2 + \cdots -)] \quad ………(2)$$

To find Residues $(R_1, R_2, \dots)$

Given the poles are

$$z^2 + 2az + 1 = 0$$

$$z = \frac{-2a \pm \sqrt{4a^2 - 4}}{2}$$

$$= \frac{-2a \pm 2\sqrt{a^2 - 1}}{2}$$

$$= -a \pm \sqrt{a^2 - 1}$$

$$\therefore \left.\begin{aligned} z_1 &= -a + \sqrt{a^2-1} \\ z_2 &= -a - \sqrt{a^2-1} \end{aligned}\right\} \text{ are simple poles}$$

$$\therefore |z_2| = -1||a + |\sqrt{a^2-1}\ |$$
$$= a + \sqrt{a^2-1}$$
$$|z_2| > 1$$

$\therefore z_2$ lies outside of c

z_1, z_2 are the roots of $z^2 + 2az + 1 = 0$

$$\therefore z_1 z_2 = \frac{1}{1} = 1$$
$$\Rightarrow z_1 z_2 = 1$$
$$\Rightarrow |z_1 z_2| = 1$$
$$\Rightarrow |z_1||z_2| = 1$$
$$\Rightarrow |z_1| = \frac{1}{|z_2|}$$
$$\Rightarrow |z_1| < 1$$

$\therefore z_1$ lies out inside of ' c '

To find R_1:

We know that,

$$R_1 = \text{Res}_{z=z_1} f(z)$$
$$= \lim_{z \to z_1} (z - z_1) f(z)$$
$$= \lim_{z \to 2_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)}$$
$$= \lim_{z \to z_1} \frac{1}{(z - z_2)}$$
$$R_1 = \frac{1}{z_1 - z_2}$$
$$R_1 = \frac{1}{2\sqrt{a^2-1}} \text{ Sub in (2)}$$

$(2) \Rightarrow$

$$I = \int_0^{\pi} \frac{d\theta}{a + \cos\theta}$$
$$I = 2\pi \left[\frac{1}{2\sqrt{a^2-1}}\right] = \frac{\pi}{\sqrt{a^2-1}}$$

2.Evaluate $\int_0^{\pi/2} \frac{dx}{a+\sin^2 x}$

Solution:

Let $\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = \int_0^{\pi/2} \frac{d\theta}{a+\left[\frac{1-\cos 2\theta}{2}\right]}$

$$= \int_0^{\pi/2} \frac{2d\theta}{2a+1-\cos 2\theta}$$

put $2\theta = \phi$

$\Rightarrow 2d\theta = d\phi$

When, $\theta = 0 \Rightarrow \phi = 0$

$\theta = \pi/2 \Rightarrow \phi = \pi$

$$\therefore \int_0^{\pi/2} \frac{2\,d\theta}{2a+1-\cos 2\theta} = \int_0^{\pi} \frac{2\,d\phi/2}{2a+1-\cos\phi}$$

$$= \int_0^{\pi} \frac{d\phi}{2a+1-\cos\phi}$$

$= \frac{1}{2}\int_0^{2\pi} \frac{d\phi}{2a+1-\cos\phi}$ .... (1)

Where $\phi = 0$ to 2π

Let $c: |z| = 1$

$z = e^{i\phi}$

$dz = e^{i\phi} i d\phi$

$$\Rightarrow d\phi = \frac{dz}{iz}$$

and also $\cos\phi = \frac{1}{2}\left[z+\frac{1}{z}\right]$

$\therefore (1) \Rightarrow$

$$\int_0^{\pi/2} \frac{dx}{a+\sin^2\theta} = \int_0^{\pi/2} \frac{2d\theta}{2a+1-\cos 2\theta}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\phi}{2a+1-\cos\theta}$$

$$= \frac{1}{2}\int_C \frac{1}{2a+1-\frac{1}{2}\left[z+\frac{1}{2}\right]} \frac{dz}{iz}$$

$$= \frac{1}{2}\int_C \frac{dz}{iz\left[\frac{4az+2z-z^2-1]}{2z}\right.}$$

$$= \frac{1}{2}\int_C \frac{2zdz}{iz[4az + 2z - z^2 - 1]}$$

$$= \frac{-1}{i}\int_C \frac{dz}{[z^2 - 2z(1 + 2a) + 1]}$$

$$= \frac{-1}{i}[2\pi i(R_1 + R_2 + \cdots -)] \quad \ldots\ldots\ldots (2)$$

To find Residues:

To poles are,

$$z^2 - 2(2a + 1)z + 1 = 0 \qquad \ldots\ldots\ldots\ldots (3)$$

$$\therefore z = \frac{2(2a + 1) \pm \sqrt{4(2a + 1^2 - 4}}{2}$$

$$= \frac{2(2a + 1) \pm 2\sqrt{(2a + 1)^2 - 1}}{2}$$

$$= (2a + 1) \pm \sqrt{4a^2 + 4a + 1 - 1}$$

$$= 2a + 1 \pm 2\sqrt{a^2 + a}$$

$$z = (2a + 1) \pm 2\sqrt{a^2 + a}$$

$\therefore$ The roots are

$$z_1 = (2a + 1) + 2\sqrt{a^2 + a}$$

$$z_2 = (2a + 1) - 2\sqrt{a^2 + a}$$

$$\therefore |z_1| > 1$$

$\therefore z_1$ lies outside of c.

We know that, (from (3))

$$z_1 z_2 = 1$$

$$|z_1 z_2| = 1$$

$$|z_1||z_2| = 1 \Rightarrow |z_2| = \frac{1}{|z_1|}$$

$$|z_2| < 1$$

$\therefore z_2$ lies inside of c

$\therefore$ To find R_2 :

$R_2 = \text{Res}_{z \to z_2} f(z)$

$$= \lim_{z \to z_2} (z - z_2) \frac{1}{(z - z_1)(z - z_2)}$$

$$= \lim_{z \to z_2} \frac{1}{(z - z_1)}$$

$$= \frac{1}{z_2 - z_1} = \frac{-1}{4\sqrt{a^2 + a}}$$

$$\therefore R_2 = \frac{-1}{4\sqrt{a^2 + a}} \quad \text{Sub in (2).}$$

$\therefore$ (2) $\Rightarrow$

$$\int_0^{\pi/2} \frac{dx}{a + \sin^2 x} = \frac{-1}{i}\left[2\pi i \left(\frac{-1}{4\sqrt{a^2 + a}}\right)\right.$$

$$= \frac{\pi}{2\sqrt{a^2 + a}}$$

3. Prove that $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 20} d\theta = \frac{3\pi}{8}$

Solution:

$$Let \;\; I = \int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta$$

$$= \int_0^{2\pi} \frac{\frac{1 + \cos 2(3\theta)}{2}}{5 - 4\cos 2\theta} d\theta$$

$$= \text{R.P} \frac{1}{2} \int_0^{2\pi} \frac{1 + e^{i6\theta}}{5 - 4\cos 2\theta} d\theta \quad \ldots\ldots\ldots.(1)$$

Take $c: |z| = 1$ (or)

$$z = e^{i\theta}, 0 < \theta < 2\pi$$

$$dz = e^{i\theta} i d\theta$$

$$\Rightarrow d\theta = \frac{dz}{iz}$$

also, $\cos\theta = \frac{1}{2}\left[z + \frac{1}{z}\right]$

And $e^{i6\theta} = \left[e^{1\theta}\right]^6 = z^6$

Let $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^2 - 1$$

$$= 2\left[\frac{1}{4}\left[z^2 + \frac{1}{z^2} + 2\right] - 1\right]$$

$$= \frac{1}{2}\left[z^2 + \frac{1}{z^2} + 2\right] - 1$$

$$=\frac{1}{2}\left[z^2+\frac{1}{z^2}\right]+1-1$$

$$=\frac{1}{2}\left[z^2+\frac{1}{z^2}\right] \quad \ldots\ldots\ldots\ldots (2)$$

$$I = \text{R.}P\frac{1}{2}\int_C \frac{1+z^6}{5-1\left[\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\right]}\frac{dz}{iz}$$

$$= R\cdot P\frac{1}{2i}\int_c \frac{1+z^6}{5-2\left[\frac{z^4+1}{z^2}\right]}\frac{dz}{z}$$

$$= R.P\frac{1}{2i}\int_C \frac{1+z^6}{\left(\frac{5z^2-2z^4-2}{z^2}\right)z}dz$$

$$= R\cdot P\frac{1}{2i}\int_C \frac{z(1+z^6)dz}{5z^2-2z^4-2}$$

$$= R\cdot P\left(\frac{-1}{2i}\right)\int_C \frac{z(1+z^6)}{2z^4-5z^2+2}\,\text{dz}$$

$$= R\cdot P\left(\frac{-1}{2i}\right) 2\pi i\,(R_1+R_2+\cdots\ldots) \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

poles are given by,

$$2z^4-5z^2+2=0$$
$$2z^2(z^2-2)-1(z^2-2)=0$$
$$(z^2-2)(2z^2-1)=0$$
$$z^2=2\,,\quad 2z^2=1$$
$$z=\pm\sqrt{2} \qquad z^2=\frac{1}{2}\Rightarrow z=\pm\frac{1}{\sqrt{2}}$$

The simple poles are

$z_1=\frac{1}{\sqrt{2}}, z_2=\frac{-1}{\sqrt{2}}$ alone lies inside of c

To find R_1, R_2 :

$$R_1 = \text{Res}\, f(z)$$
$$z=\frac{1}{\sqrt{2}}$$
$$=\lim_{z\to\frac{1}{\sqrt{2}}}\frac{P(z)}{Q'(z)}$$

[where $P(z)=z+z^7$

$Q(z) = 2z^4 - 5z^2 + 2$

$$= \lim_{z \to \frac{1}{\sqrt{2}}} \frac{z + z^7}{8z^3 - 10z}$$

$$= \lim_{z \to \frac{1}{\sqrt{2}}} \frac{z(1 + z^6)}{2z(4z^2 - 5)}$$

$$= \lim_{z \to 1/\sqrt{2}} \frac{(1 + z^6)}{2(4z^2 - 5)}$$

$$= \frac{1 + (1/\sqrt{2})^6}{2\left[4\left(\frac{1}{\sqrt{2}}\right)^2 - 5\right]}$$

$$= \frac{1 + \frac{1}{2}}{8\left(\frac{1}{2}\right) - 10} = \frac{9/8}{-6}$$

$$R_1 = \frac{-3}{16}$$

Similarly, $R_2 = -\frac{3}{16}$ sub in(3)

$(3) \Longrightarrow I = R.P\left(\frac{-1}{2i}\right) 2\pi i \left[\frac{-3}{16} - \frac{-3}{16}\right]$

$$= \mathrm{R} \cdot \mathrm{P}\left[\pi\left(\frac{2(3)}{16}\right)\right]$$

$$= \mathrm{R} \cdot \mathrm{P}\left(\frac{3\pi}{8}\right)$$

$$= \frac{3\pi}{8}$$

Type: II

Integral of the form $\int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\, dx$, where

(i) deg $Q(x) \geq$ deg $P(x) + 2$

(ii) $Q(x) \neq 0$ for any real x.

(iii) $P(x)$ and $Q(x)$ are polynomial in x.

Proof:

Taking the contour $c: \Gamma \cup L$

$\Gamma: |z| = R$

$$\int_C \frac{P(z)}{Q(z)} dz = \int_{\Gamma U L} \frac{P(z)}{Q(z)} dz$$

$$= \int_\Gamma \frac{P(z)}{Q(z)} dz + \int_L \frac{P(z)}{Q(z)} dz$$

$$= \int_\Gamma \frac{P(z)}{Q(z)} dz + \int_{-R}^{R} \frac{P(z)}{Q(z)} dz$$

Taking $\lim_{R\to\infty}$,

$$\int_c \frac{P(z)}{Q(z)} dz = 0 + \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx$$

$$= 2\pi i[R_1 + R_2 + \cdots] \quad \ldots\ldots\ldots..(1)$$

Result 1:

$$\lim_{z\to\infty} zf(z) = 0$$

$$\Rightarrow \lim_{R\to\infty} \int_H f(z)dz = 0$$

Result 2:

$$\lim_{R\to\infty} \int_\Gamma \frac{P(z)}{Q(z)} dz = 0$$

Since degree of $Q(z) \geq$ deg of $P(z) + 2$

Result 3:

$$\int_{-\infty}^{\infty} f(x)dx = \begin{cases} 2\int_0^{\infty} f(x)dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

Problem:

1.Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9} dx$

Solution:

Take the contour $C: \Gamma U L$

Where $\Gamma: |z| = R$

and $L: [-R, R]$.

Consider,

$$\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = \int_\Gamma \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz + \int_L \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$$

$$= 2\pi i[R_1 + R_2 + \cdots] \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} \frac{z[z^2 - z + 2]}{z^4 + 10z^2 + 9}$$

$$= \lim_{z\to\infty} \frac{z\,z^2[1 - \frac{1}{z} + \frac{1}{z^2}]}{z^4[1 + \frac{10}{z^2} + \frac{9}{z^4}]}$$

=0

$\lim_{R\to\infty} \int_\Gamma \frac{z^2-z+2}{z^4+10z^2+9} dz = 0$

Taking $\lim_{R\to\infty}$ in (1) and using (2).

$\therefore (1) \Rightarrow$

$$\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = 0 + \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$$

$$= 2\pi i[R_1 + R_2 + \cdots] \quad \ldots\ldots\ldots\ldots \ (2)$$

poles are given by

$$x^4 + 10x^2 + 9 = 0$$
$$x^4 + x^2 + 9x^2 + 9 = 0$$
$$x^2(x^2 + 1) + 9(x^2 + 1) = 0$$
$$(x^2 + 9)(x^2 + 1) = 0$$
$$(x + i)(x - i)(x + 3i)(x - 3i) = 0$$

$\therefore x = \pm 3i, \pm i$ are Simple poles.

Among these poles, only $x = i$ and $3i$ are lie inside ' c '.

To find R_1 & R_2:

$$\therefore R_1 = \text{Res}_{z=i}^{f(z)} = \lim_{z\to i} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to i} \frac{z^2 - z + 2}{4z^3 + 20z}$$

$$= \frac{i^2 - i + 2}{4i^3 + 20i} =$$

$$= \frac{-1 - i + 2}{-4i + 20i}$$

$$R_1 = \frac{1 - i}{16i}$$

Similarly, $R_2 = \text{Res}_{z\to 3i} f(z) = \lim_{z\to 3i} \frac{P(z)}{Q'(z)}$

$$R_2 = \frac{7 + 3i}{48i}$$

Sub R_1 & R_2 in (3)

$\therefore$ (3) $\Rightarrow$

$$\int_C \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = 2\pi i \left[\frac{1-i}{16i} + \frac{7+3i}{48i}\right]$$

$$= 2\pi i \left[\frac{3 - 3i + 7 + 3i}{48i}\right]$$

$$= \pi \left[\frac{10}{24}\right]$$

$$= \frac{5\pi}{12}.$$

2.Evaluate $\int_0^\infty \frac{x^2 dx}{x^4+5x^2+6}$

Solution:

Let $c: \Gamma \cup L$

where $\Gamma: |z| = R$ & $L = [-R, R]$

Consider,

$$\int_c \frac{z^2}{z^4 + 5z^2 + 6} dz = \int_\Gamma \frac{z^2}{z^4 + 5x^2 + 6} dz \int_{-R}^{R} \frac{z^2}{z^4 + 5z^2 + 6} dz$$

$$= 2\pi i [R_1 + R_2] \qquad \ldots\ldots\ldots.(1)$$

Taking $\lim_{R\to\infty}$ in (1)

$$\therefore (1) \Rightarrow \int_C \frac{z^2}{z^4 + 5z^2 + 6} dz = 0 + \int_{-\infty}^{\infty} \frac{x^2}{x^4 + 5x^2 + 6} dx$$

$=2\pi i[R_1 + R_2 + \cdots]$

$\left[\lim_{z\to\infty} zf(z) = 0 \right.$

$\left. \Rightarrow \lim_{R\to\infty} \int_r \frac{P(z)}{Q(z)} dz = 0 \right]$

$\therefore$ poles are given by

$$x^4 + 5x^2 + 6 = 0$$
$$x^4 + 3x^2 + 2x^2 + 6 = 0$$
$$x^2[x^2 + 3] + 2[x^2 + 3] = 0$$
$$(x^2 + 3)(x^2 + 2) = 0$$
$$x^2 = -2, x^2 = -3$$
$$x = \pm\sqrt{2}i, x = \pm\sqrt{3}i$$

are simple poles.

Among these poles, $x = \sqrt{2i}$ & $\sqrt{3}i$ only lies inside c.

To find R_1 & R_2 :

$$R_1 = \text{Res}_{z\to\sqrt{2}i} f(z)$$

$$= \lim_{z\to\sqrt{2}i} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to\sqrt{2}i} \frac{z^2}{4z^3+10z}$$

$$= \frac{(\sqrt{2}i)^2}{4(\sqrt{2}i)^3+10(\sqrt{2}i)}$$

$$= \frac{\sqrt{2}i\sqrt{2}i}{\sqrt{2}i[4(\sqrt{2})^2+10]}$$

$$= \frac{\sqrt{2}i}{-8+10}$$

$$R_1 = \frac{\sqrt{2}i}{2}$$

$$Similarly,\ R_2 = \frac{-\sqrt{3}i}{2}$$

∴ Sub R_1 & R_2 in (2)

$$\therefore \int_{-\infty}^{\infty} \frac{z^2}{z^4+5z^2+6} dz = 2\pi i\left[\frac{\sqrt{2}i-\sqrt{3}i}{2}\right]$$

$$= \pi i^2[\sqrt{2}-\sqrt{3}]$$

$$= -\pi[\sqrt{2}-\sqrt{3}]$$

$$= \pi[\sqrt{3}-\sqrt{2}]$$

$$\therefore \int_{0}^{\infty} \frac{z^2}{z^4+5z^2+6} dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{z^2}{z^4+5z^2+6} dz$$

$$= \frac{1}{2}[\pi\sqrt{3}-\pi\sqrt{2}]$$

$$= \frac{\pi}{2}[\sqrt{3}-\sqrt{2}]$$

3. Evaluate $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$

Solution:

Let $C: \Gamma UL$

where $\Gamma: |z| = R$ & $L = [-R, R]$

Consider,

$$\int_c \frac{dz}{z^4+1} = \int_\Gamma \frac{dz}{z^4+1} + \int_{-R}^{R} \frac{dz}{z^4+1}$$

$= 2\pi i[R_1 + R_2 + \cdots ..]$ .... (1)

Taking $\lim_{R\to\infty}$ in (1)

(1) $\Rightarrow$

$$\int_c \frac{dz}{z^4+1} = 0 + \int_{-\infty}^{\infty} \frac{dx}{z^4+1}$$

$= 2\pi i[R_1 + R_2 + \cdots ..]$ .... (2)

poles are given by

$x^4 + 1 \Rightarrow x^4 = -1$

$x^4 = \cos\pi$

Generally,

$z^4 = \cos[2k + \pi\pi$

$z^4 = e^{i[2k+1]\pi}, k = 0,1,2$

$z = e^{i\frac{(2k+1)\pi}{4}}$

$\therefore z = e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i\pi/4}$

$Z = e^{i\pi/4} \& e^{i3\pi/4}$

alone lies inside c.

To find R_1 & R_2

$$R_1 = \operatorname*{Res}_{z=e^{i\pi/4}} f(z) = \frac{1}{4[e^{i\pi/4}]^3}$$

$$= \frac{1}{4[e^{3i\pi/4}]}$$

$$= \frac{1}{4}\left[e^{-i3\pi/4}\right]$$

$$= \frac{1}{4}[\cos 3\pi/4 - i\sin 3\pi/4]$$

$$= \frac{1}{4}[\cos 135° - i\sin 135°]$$

$$= \frac{1}{4}[\cos(180° - 45°) - i\sin(180° - 45°)]$$

$$= \frac{1}{4}[-\cos 45° - i\sin 45]$$

$$= \frac{1}{4}\left[-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$$

$$R_1 = \frac{-1}{4\sqrt{2}}[1+i]$$

Similarly,

$$R_2 = \text{Res}_{z\to e^{3i\pi/4}} f(z)$$
$$= \lim_{z\to e^{3i\pi/4}} \frac{1}{4z^3}$$
$$= \frac{1}{4[e^{3i\pi/4}]^3}$$

$$= \frac{1}{4[e^{9i\pi/4}]}$$
$$= \frac{1}{4}e^{-9i\pi/4} = \frac{1}{4}e^{-i[8\pi/4+\pi/4]}$$
$$= \frac{1}{4}[\cos(360+45) - \sin(360+45)]$$
$$= \frac{1}{4}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$
$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$$
$$R_2 = \frac{1}{4\sqrt{2}}[1-i]$$

Sub R_1 & R_2 in eqn (2)

$$(2) \Rightarrow \int_{-\infty}^{\infty} \frac{dz}{x^4+1} = 2\pi i\left[\frac{-1}{4\sqrt{2}}(1+i) + \frac{1}{4\sqrt{2}}(1-i)\right]$$
$$= \frac{2\pi i}{4\sqrt{2}}[-x - i + x - i]$$
$$= \frac{\pi i}{2\sqrt{2}}[-2i]$$
$$= \frac{\pi}{\sqrt{2}}$$

We know that,

$$\int_0^{\infty} \frac{1}{z^4+1}dz = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{z^2+1}dz$$
$$= \frac{1}{2}\left(\frac{\pi}{\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}}.$$

4.Prove that $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}dx = \frac{-\pi}{e}\sin z$

Solution:

Take the contour $C: \Gamma UL$

where $\Gamma: |z| = R$

& $L: [-R, R]$

consider,

$$\int_C \frac{\sin z}{z^2+4z+5}dz = \int_\Gamma \frac{\sin z}{z^2+4z+5}dz + \int_L \frac{\sin z}{z^2+4z+5}dz$$

$$\lim_{z\to\infty} = \int \frac{Ime^{ix}}{x^2+4x+5}dx$$

$$= I_m \int \frac{e^{ix}}{x^2+4x+5}dx$$

poles are given by

$$= \frac{-4 \pm \sqrt{16-4\times 5\times 1}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm \sqrt{4i^2}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$$= (-2+i)(-2-i)$$

Among these poles $x = (-2+i)$ only lies inside c.

To find $R_1 \& R_2$:-

$$R_1 = \text{Res}_{z\to(-2+i)} f(z)$$

$$= \lim_{z\to(-2+i)} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to(-2+i)} \frac{e^i z}{2z+4}$$

$$= \frac{e^{i(-2+i)}}{2(-2+i)+4}$$

$$= \frac{e^{-2i+i^2}}{2i+2i+4}$$

$$= \frac{e^{-1-2i}}{2i}$$

$\therefore (1) \Rightarrow$

$$= 2\pi\dot{x}\left[\frac{e^{-1}\cdot e^{-2i}}{2\text{i}}\right]$$
$$= \frac{\pi}{e}e^{-2i}$$
$$= \text{Im}\frac{\pi}{e}[\cos 2 - i\sin 2]$$

Taking only the imaginary part,

$$= \frac{\pi}{e}[-\sin 2]$$
$$= \frac{-\pi}{e}\sin 2.$$

Type III:

Integral of the form $\int_{-\infty}^{\infty} \sin mx f(x)dx$ (or) $\int_{-\infty}^{\infty} \cos mx f(x)dx$. where (i) $m \geq 0$

(ii) $\lim_{z\to\infty} f(z) = 0$, (iii) $f(z)$ does not have poles on real axis.

proof:

Let $c = \Gamma \cup L$

$\Gamma: |z| = R$

$=$ upper semicircle

$L = [-R, R]$

Consider, $\int_C e^{imz} f(z)dz = 2\pi i[R_1 + R_2 + \cdots]$

Jordan's Lemma:

$$\lim_{z\to\infty} f(z) = 0 \Rightarrow \lim_{R\to\infty} \int_\Gamma e^{imz} f(z)dz = 0$$

where $\Gamma: |z| = R =$ upper Semicircle.

Problem 1:

1.Evaluate (a) $\int_a^x \frac{\cos x}{x^2+a^2} dx$, where a real

(b) $\int_0^\infty \frac{\cos mx}{x^2+a^2} dx = \frac{\pi e^{-ma}}{2a}$ where $\begin{matrix} m > 0 \\ a > 0, \end{matrix}$

(c) $\int_0^\infty \frac{\cos mx}{x^2+1} dx = \frac{\pi e^{-m}}{2}$ where $m > 0$.

Proof:

Let $c = \Gamma \cup L$

Where $\Gamma: |z| = R$ (upper Semicircle)

$L = [-R, R]$

Consider,

$\int_C \frac{e^{imz}}{z^2+a^2} dz = \int_\Gamma \frac{e^{imz}}{z^2+a^2} dz + \int_{-R}^{R} \frac{e^{imx}}{x^2+a^2} dx = 2\pi i[R_1 + R_2 + \cdots]$ ………… (1)

poles are given by,

$$z^2 + a^2 = 0$$
$$\Rightarrow z^2 = -a^2$$
$$\Rightarrow z = \pm ai$$

Take $a > 0$,

∴ The simple pole $z = i$ a alone lies inside' c

[where $a < 0$, The Simple pole z =-ai alone lies inside ' c '].

To find R:

We know that,

$$R_1 = \text{Res} f(z)$$
$$= \lim_{z \to a_i} \frac{P(z)}{Q'(z)}$$
$$= \lim_{z \to a_i} \frac{e^{imz}}{2z}$$
$$= \frac{e^{imai}}{2ai} = \frac{e^{mai^2}}{2ai} = \frac{e^{-ma}}{2ai} \ldots \ldots \ldots . (2).$$

clearly,

$$\lim_{z \to \infty} \frac{1}{z^2 + a^2} = 0$$

$$\lim_{R \to \infty} \int_\Gamma \frac{e^{imz}}{z^2 + a^2} dz = 0 \quad \ldots \ldots \ldots \ldots \ldots \ldots . (3)$$

(by Jordan's Lemma)

Sab(2) in (1),

$$\therefore \int_\Gamma \frac{e^{imz}}{z^2 + a^2} dz + \int_{-R}^{R} \frac{e^{imx}}{x^2 + a^2} dx = 2\pi i \left[\frac{e^{-ma}}{2a\dot{x}}\right]$$
$$= \pi \left[\frac{e^{-ma}}{a}\right]$$

Taking lim & using $R \to \infty$ in equation (3)

∴ We have. $0 + \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x^2+a^2} dx = \frac{\pi}{a} e^{-ma}$

(b) equating real parts in (4)

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{\pi}{a} e^{-ma}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\cos mx}{x^2+a^2} dx = \frac{\pi}{a} e^{-ma}$$

To get (a)

put $m = 1$ in (5)

$\therefore \int_0^{\infty} \frac{\cos x}{x^2+a^2} dx = \frac{\pi}{2a} e^{-\text{ma}}$ .... (5)

To get (a)

put $a = 1$ in (5)

$\therefore$ (5) $\Rightarrow$

$$\int_{0}^{\infty} \frac{\cos mx}{x^2+1} dx = \frac{\pi}{2} e^{-m}$$

Note:

Equating imaginary parts in (4) we get.

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x^2+a^2} dx = 0.$$

Problem 2:

Evaluate (a) $\int_0^{\infty} \frac{x\sin mx}{x^2+a^2} dx = \frac{\pi}{2} e^{-a}$. where $a > 0$

(b) $\int_0^{\infty} \frac{x\sin ax}{x^2+4} dx = \frac{\pi}{2} e^{-2a}$.

Proof:

Let $c = \Gamma UL$

Where $\Gamma : |z| = R$ (Semi circle)

$L = [-R, R]$

consider,

$$\int_C \frac{z\sin zmz}{z^2+a^2} dz = \int_\Gamma \frac{z\sin mz}{z^2+a^2} dz + \int_{-R}^{R} \frac{x\sin mx}{x^2+a^2} dx$$

$= 2\pi i[R_1 + R_2 + \cdots]$ .... (1)

poles are given by,

$z^2 + a^2 = 0$

$z^2 = -a^2$

$z = \pm \text{a}\, i$

Take, $a > 0$

$\therefore$ The simple poles $z = ai$ alone lies inside ' c '.

To find R_1 $\because$

$$R_1 = \operatorname*{Res}_{z=a_i} f(z)$$

$$= \lim_{z \to a_i} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z \to a_i} \frac{ze^{imz}}{2z}$$

$$= \lim_{z \to a_i} \frac{ze^{imz}}{2z}$$

$$= \frac{aie^{imai}}{2ai}$$

$$= \frac{aie^{-ma}}{2ai}$$

Clearly,

$$\lim_{z \to \infty} \frac{z}{z^2 + a^2} dz = 0.$$

By Jordan is Lemma, $\Rightarrow \lim_{z \to \infty} \int_\Gamma \frac{ze^{imz}}{z^2+a^2} dz = 0$ ……….(3)

Sub (2) in (1)

$$\int_\Gamma \frac{e^{imz}}{z^2 + a^2} dz + \int_{-R}^{R} \frac{xe^{imx}}{x^2 + a^2} dx = \pi i \left[\frac{e^{-ma}}{\psi}\right]$$

$$= \pi i[e^{-ma}]$$

Taking lim & using $R \to \infty$ in eq n (3).

We have, $0 + \int_{-\infty}^{\infty} \frac{x(\cos mx + i\sin mx)}{x^2+a^2} dx = \pi i[e^{-ma}]$

Equating imaginary parts,

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2} dx = \pi e^{-ma}$$

$$\int_0^{\infty} \frac{x\sin mx}{x^2+a^2} dx = \frac{\pi}{2} e^{-ma} \quad \ldots\ldots\ldots\ldots (4)$$

To get (a): put $m = 1$ in (4),

$$\int_0^{\infty} \frac{x\sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$$

To get (b):- put $m = a, a = 2$ in (4)

and $\int_0^{\infty} \frac{x\sin ax}{x^2+4} dx = \frac{\pi}{2} e^{-2a}$

(3) $\int_{-\infty}^{\infty} \frac{\cos x dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$, where $a >$ b > 0

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+)(x^2+)} dx = \frac{\pi}{3}\left[\frac{1}{e} - \frac{1}{e^2}\right]$$

$$\int_{0}^{\infty} \frac{\cos mx}{(x^2+a^2)^2} dx = \frac{\pi}{4a^3}(1+ma)e^{-ma}, \text{ where } m > 0, a > 0.$$

Type-1V:

Integral of the form $\int_{-\infty}^{\infty} x^n f(x)dx$ where $f(z)$ has finite number of poles on the real axis

Let $z = a$ be a simple pole of $f(z)$ withe residue k.

Let $\widehat{AB}: |z-a| = r, \alpha \leqslant \arg(z-a) \leq \beta$

Then $\lim_{\gamma\to 0}\int_{\widehat{AB}} f(z)dz = ik(\beta - \alpha)$.

Problems:

1.Prove that $\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}\int_{-\infty}^{\infty} x^n f(x)dx$

proof:

consider $\int_C \frac{e^{iz}}{z} dz$

poles are given by, $z = 0$.

which is a simple pole lying on real axis.

$C = \Gamma \cup L_1 \cup \gamma \cup L_2$

where $\Gamma: |z| = R$ in the upper Semi-circle (half-plane)

$\gamma: |z| = r$ in the upper half-plane

$L_1 = [-R, -r]$

$L_2 = [r, R]$.

Clearly $\frac{e^{iz}}{z}$ is analytic inside and on c

By Cauchy's Theorem.

$$\int_c \frac{e^{1z}}{z} dz = 0$$

$$\int_r + \int_{L_1} + \int_\gamma + \int_{L_2} = 0$$

(i.e.,) $\int_r \frac{e^{iz}}{z} dz + \int_{-R}^{-r} \frac{e^{ix}}{x} dx - \int_\gamma \frac{e^{iz}}{z} dz + \int_\gamma^R \frac{e^{ix}}{x} dz = 0$ ………….(1)

By Jordan's Lemma,

$\lim\limits_{z\to\infty} \frac{1}{z} = 0 \Rightarrow \lim\limits_{R\to\infty} \int_\Gamma \frac{e^{iz}}{z} dz = 0$ .... (2)

$$k = \text{Res}_{z=0} \frac{e^{iz}}{z} = \lim_{z\to\infty} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to\infty} \frac{e^{iz}}{1} = \frac{e^{i0}}{1} = 1$$

(i.e.,) $K = 1$,

By Result,

$$\lim_{\gamma\to 0} \int_\gamma \frac{e^{iz}}{z} dz = ik(\beta - \alpha)$$

$$= i\ (1)\ (\pi - 0)$$

$$= i\ \pi \qquad \text{............(3)}$$

Taking $\lim\limits_{R\to\infty}$ and $r \to 0$ in (1) and using (2) & (3) we get,

$$=0 + \int_{-\infty}^{0} \frac{e^{ix}}{x} dx - i\pi + \int_0^\infty \frac{e^{ix}}{x} dx = 0$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x} dx = i\pi$$

Equating Imaginary parts.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x} dx = \ddot{d}x$$

$$2\int_0^\infty \frac{\sin x}{x} dx = \pi$$

$$\therefore \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

(2) Prove that (a) $\int_0^\infty \frac{x^{p-1}}{1-x} dx = \pi \cot p\pi\ (0 < p < 1)$

(b) $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}\ (0 < p < 1)$

Proof:

consider $\displaystyle\int_0^\infty \frac{z^{p-1}}{1-z} dz$

The poles are $z = 0$ and $1 - z = 0$.

(i.e.,) $z=0$ and $z=1$ are simple poles which lie on the real axis.

$c=\Gamma \cup_{L_1} \cup \gamma_1 \cup L_2 \cup \gamma_2\ UL_3$

$\frac{z^{P-1}}{1-Z}$ is analytic inside and on C.

∴ By Cauchy's theorem,

$$\int_C \frac{z^{p-1}}{1-z} dz = 0$$

$$\int_\Gamma + \int_{L_1} + \int_{\gamma_1} + \int_{L_2} + \int_{\gamma_2} + \int_{L_3} = 0$$

$$\text{(ie)} \int_\Gamma \frac{z^{p-1}}{1-z} dz + \int_{-R}^{-r_1} \frac{x^{p-1}}{1-x} dx - \int_{-r_1}^{r_1} \frac{z^{p-1}}{1-z} dz$$

$$+ \int_{r_1}^{1-r_2} \frac{x^{p-1}}{1-x} dx - \int_{1-r_2}^{1+r_2} \frac{z^{p-1}}{1-z} dz + \int_{1+\gamma_2}^{R} \frac{x^{p-1}}{1-x} dx = 0$$

By Lemma,

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty} z \cdot \frac{z^{p-1}}{1-z}$$

$$= \lim_{z\to\infty} \frac{z^p}{1-z}$$

$$= 0\ (0 < p < 1)$$

$$\Rightarrow \lim_{R\to\infty} \int_\Gamma f(z) dz = 0$$

$$\Rightarrow \lim_{R\to\infty} \int_\Gamma \frac{z^{p-1}}{1-z} dz = 0$$

By result

$$\lim_{\gamma_1\to 0} \int_{\gamma_1^+} \frac{z^{p-1}}{1-z} dz = ik, (\beta - \alpha)$$

$$k_1 = \text{Res} \frac{z^{p-1}}{1-z}$$

$$= \lim_{z\to 0} \frac{z^{p-1}}{x^{p-1}} \left(\because \frac{p(z)}{Q'(z)} \right)$$

$$k_1 = 0, \quad \lim_{\gamma_1\to 0} \int \frac{z^{p-1}}{1-z} dz = 0 \ldots\ldots\ldots\ldots (2)$$

$$\lim_{\gamma_2\to 0} \int \frac{z^{p-1}}{1-z} dz = ik_2(\beta - \alpha)$$

$$k_2 = \text{Res}\frac{z^{P-1}}{1-z}$$

$$= \lim_{z\to 1}(z-1)\frac{z^{P-1}}{1-z}$$

$$= (-1)1^{P-1}$$

$$k_2 = -1$$

$$\lim_{r_2\to 0}\int \frac{z^{P-1}}{1-2}dz = i(-1)(\pi-0) = -\pi i \quad \ldots\ldots\ldots\ldots(3)$$

Taking $\lim_{R\to\infty}$ and $r_1 \to 0$ and $r_2 \to 0$ in (1) and using (2) and (3) we get,

$$0 + \int_{-\infty}^{0}\frac{x^{p-1}}{1-x}dx - 0 + \int_{0}^{1}\frac{x^{p-1}}{1-x}dx - (-i\pi) + \int_{1}^{\infty}\frac{x^{p-1}}{1-x}dx = 0$$

$$\Rightarrow \int_{-\infty}^{0}\frac{x^{p-1}}{1-x}dx + \int_{0}^{\infty}\frac{x^{p-1}}{1-x}dx = -i\pi \rightarrow \text{(-4)}$$

$$I_1 + I_2 = -i\pi \rightarrow (4)$$

In I_1 : put $x = -x$

$$dx = -dx$$

$$I_1 = \int_{-\infty}^{0}\frac{(-x)^{p-1}}{1+x}(-dx)$$

$$= \int_{0}^{\infty}\frac{(-1)^{p-1}(x)^{p-1}}{1+x}dx$$

(i.e.,) $I_1 = \int_0^{\infty}\frac{(-1)^{P-1}(x)^{P-1}}{1+x}dx$

Sub in (4)

$$\int_{0}^{\infty}\frac{(-1)^{P-1}x^{P-1}}{1+x}dx + \int_{0}^{\infty}\frac{x^{P-1}}{1-x}dx = -i\pi$$

multiply $(-1) \Rightarrow (-1)^P I_3 - I_2 = i\pi$

where $I_3 = \int_0^{\infty}\frac{x^{p-1}}{1+x}dx$

$$(\cos\pi + \text{i}\sin\pi)^P I_3 - I_2 = i\pi$$

$(\cos \text{p}\pi + i\sin \text{p}\pi)I_3 - I_2 = i\pi$ ……….. (5)

Equating Imaginary part and Real part in (5)

$I.P: \sin P\pi I_3$, $\quad \therefore I_3 = \dfrac{\pi}{\sin p\pi}$ ....(6)

R.P: $\cos P\pi I_3 - I_2 = 0$

$$\text{cosp}\, p\pi \cdot \frac{\pi}{\sin p\pi} = I_3$$

$$\therefore \int_0^{\infty} \frac{x^{p-1}}{1-x} dx = \pi \cot p\pi$$

5.Evaluate $\int_0^{\infty} \frac{\log}{1+x^2} dx$

Proof:

consider $f(x) = \frac{\log z}{1+z^2}$

poles are given by,

$1 + z^2 = 0$

$z^2 = -1$

$z = Ii$ are simple poles.

$z = 0$ is a branch point of $\log z$

Let $c = \Gamma \cup L_1 \cup \gamma \cup L_2$

$z = i$ alone lies inside c

$$R_1 = \text{Res}_{z=i} f(z) = \lim_{z\to i} \frac{P(z)}{Q'(z)}$$

$$= \lim_{z\to i} \frac{\log z}{2z}$$

$$= \frac{\log i}{2i}$$

$$= \frac{\log e^{i\pi/2}}{2i} \quad [i = \cos \pi/2 + i\sin \pi/2 = e^{i\pi/2}]$$

$$= \frac{i\pi/2 \log \mathrm{e}}{2i}$$

$$R_1 = \pi/4$$

By Residue Theorem,

$$\int_C f(z)dz = 2\pi i(R_1 + R_2 + \cdots)$$

$$\int_{\Gamma} + \int_{L_1} + \int_{\gamma} + \int_{L_2} = 2\pi i(\pi/4) = \frac{\pi^2 i}{2}$$

$$\int_{\Gamma} \frac{\log z}{1+z^2} dz + \int_{-\mathrm{R}}^{-\mathrm{r}} \frac{\log x}{1+x^2} dx - \int_{\gamma} \frac{\log z}{1+z^2} dz + \int_{\mathrm{r}}^{\mathrm{R}} \frac{\log x}{1+x^2} dx = \frac{i\pi^2}{2} \quad \text{..........} (1)$$

Where $\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} z \frac{\log z}{1+z^2}$

$$= \lim_{z\to\infty} \frac{\log z}{z} \frac{z^2}{1+z^2}$$

$$= \lim_{z\to\infty} \frac{\log z}{z} \lim_{z\to\infty} \frac{z^2}{1+z^2}$$

$$= \lim_{z\to\infty} \frac{\log z}{z} \lim_{z\to\infty} \frac{z^2}{z^2(1+1/z^2)}$$

$$= \lim_{z\to\infty} \frac{\log z}{z} \frac{1}{0+1}$$

$$= \lim_{z\to\infty} \frac{1/z}{1}$$

$$\lim_{z\to\infty} z\, f(z) = 0$$

$$\lim_{R\to\infty} \int_\Gamma \frac{\log z}{1+z^2}\, dz = 0 \qquad \ldots\ldots\ldots\ldots (2)$$

$$\lim_{r\to 0} \int_\gamma \frac{\log z}{1+z^2}\, dz = ik\,(\beta - \alpha)$$

$$= ik\,(\pi - 0) = ik\pi \qquad \ldots\ldots\ldots\ldots (3)$$

$where\ k = \operatorname*{Res}_{z=0} f(z)$

$$= \lim_{z\to 0} (z-0) f(z)$$

$$= \lim_{z\to 0} z \cdot \frac{\log z}{1+z^2}$$

put $z = 1/t$, as $z \to 0 \Rightarrow t \to \infty$

$$k = \lim_{t\to\infty} \frac{1}{t} \frac{\log(1/t)}{1+1/t^2}$$

$$= \lim_{t\to\infty} \frac{1}{t} \frac{\log 1 - \log t}{\left(\frac{t^2+1}{t^2}\right)}$$

$$= -\lim_{t\to\infty} t \frac{\log t}{t^2+1}$$

$$= -\lim_{t\to\infty} t \cdot \frac{\log t}{t^2+1}$$

$k = -0$ (as in (A))

Sub in (3) $\lim_{r\to 0} \int_\gamma \frac{\log z}{1+z^2} dz = 0 \qquad \ldots\ldots\ldots\ldots(4)$

Taking $\lim R \to \infty, r \to 0$ in (1) and using (2) and (1)

$$0+\int_{-\infty}^{0}\frac{\log x}{1+x^2}dx-0+\int_{0}^{\infty}\frac{\log x}{1+x^2}dx=\frac{i\pi^2}{2}$$

put $x=-t \Rightarrow dx=-dt$

$$\int_{0}^{\infty}\frac{\log(-t)}{1+t^2}(-dt)+\int_{0}^{\infty}\frac{\log x}{1+x^2}dx=\frac{i\pi^2}{2}$$

$$\int_{0}^{\infty}\frac{\log(-1)+\log t}{1+t^2}dt+\int_{0}^{\infty}\frac{\log x}{1+x^2}dx=\frac{i\pi^2}{2}$$

$$\int_{0}^{\infty}\frac{\log e^{i\pi}}{1+t^2}dt+\int_{0}^{\infty}\frac{\log t}{1+t^2}dt+\int_{0}^{\infty}\frac{\log x}{1+x^2}dx=\frac{i\pi^2}{2}.$$

$$\int_{0}^{\infty}\frac{i\pi}{1+t^2}dt+2\int_{0}^{\infty}\frac{\log x}{1+x^2}dx=\frac{i\pi^2}{2}$$

$$R\cdot P \therefore 2\int_{0}^{\infty}\frac{\log x}{1+x^2}dx=0$$

$$\therefore \int_{0}^{\infty}\frac{\log x}{1+x^2}dx=0 \qquad \ldots\ldots\ldots\ldots(5)$$

$$I.P \therefore \pi\int_{0}^{\infty}\frac{dx}{1+x^2}=\pi^2/2$$

$$\therefore \int_{0}^{\infty}\frac{dx}{1+x^2}=\pi/2$$

6. Evaluate $\int_0^\infty \frac{x^{1/3}}{1+x^2}dx$

Proof:

consider $f(z)=\frac{z^{1/3}}{1+z^2}=\frac{P(z)}{Q(z)}$

poles are given by,

$1+z^2=0$

$z^2=-1$

$z=\pm i$ are simple poles

$z^{1/3}$ is a inane valued function and $z=0$ is a branch point.

$C=\Gamma\cup L_1\cup\gamma\cup L_2$

The simple pole $z=i$ alone lies inside C

$$R_1 = \text{Res}_{z=i} f(z)$$
$$= \lim_{z\to i} \frac{z^{1/3}}{1+z^2}$$
$$= \lim_{z\to i} \frac{z^{1/3}}{2z} = \frac{p(z)}{2'(z)}$$
$$= \frac{i^{1/3}}{2i}$$
$$= \frac{\left(e^{i\pi/2}\right)^{1/3}}{2i}$$
$$R_1 = \frac{e^{i\pi/6}}{2i}$$

By Residue Theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{e^{i\pi/6}}{2i}\right)$$
$$= \pi e^{i\pi/6} \ldots\ldots\ldots\ldots (1)$$

$$\lim_{z\to\infty} \mathrm{z\, f(z)} = \lim_{z\to\infty} \mathrm{z}\frac{z^{1/3}}{1+z^2}$$
$$= \lim_{z\to\infty} \mathrm{z}\frac{z^{4/3}}{z^2(1+\frac{1}{z^2})}$$
$$= \lim_{z\to\infty} \mathrm{z}\frac{z^{4/3}}{z^{2/3}(1+\frac{1}{z^2})} = 0$$

$$\lim_{R\to\infty} \mathrm{z}\frac{z^{1/3}}{1+z^2}dz = 0 \quad \ldots\ldots\ldots\ldots (2)$$

$$\lim_{\gamma\to 0} \int_\gamma f(z)dz = \mathrm{ik}\ (\beta - \alpha)$$

$$= \mathrm{ik}\ (\pi - 0) = = \mathrm{ik}\,\pi \qquad \ldots\ldots\ldots\ldots(3)$$

Where $k = \text{Res}_{z=0} f(z)$

$$= \lim_{z\to 0}(z-0)\frac{z^{1/3}}{1+z^2}$$
$$= \frac{0}{1+0} = 0$$

Sub in (3), $\lim_{\gamma\to 0} \int_\gamma \frac{z^{1/3}}{1+z^2}dz = 0$ ....(4)

Taking $\lim R \to \infty, r \to 0$ in (1) and using (2) & (4), we get,

$$0+\int_{-\infty}^{0}\frac{x^{1/3}}{1+x^2}dx-0+\int_{0}^{\infty}\frac{x^{1/3}}{1+x^2}dx=\pi e^{i\pi/6}$$
$$\int_{-\infty}^{0}\frac{x^{1/3}}{1+x^2}dx+\int_{0}^{\infty}\frac{x^{1/3}}{1+x^2}dx=\pi e^{i\pi/6}$$

Put $x=-t \Rightarrow dx=-dt$

$$\int_{0}^{\infty}\frac{(-t)^{1/3}}{1+(-t)^2}(-dt)+\int_{0}^{\infty}\frac{x^{1/3}}{1+x^2}dx=\pi e^{i\pi/6}$$
$$\int_{0}^{\infty}\frac{(-1)^{1/3}t^{1/3}}{1+t^2}dt+\int_{0}^{\infty}\frac{x^{1/3}}{1+x^2}dx=\pi e^{i\pi/6}$$
$$\int_{0}^{\infty}\frac{x^{1/3}}{1+x^2}\left[(-1)^{1/3}+1\right]dx=\pi e^{i\pi/6}$$
$$I\left[\left(e^{i\pi}\right)^{1/3}+1\right]=\pi e^{i\pi/6}$$
$$I\left[e^{i\pi/3}+1\right]=\pi e^{i\pi/6}$$
$$I[\cos\pi/3+i\sin\pi/3]=\pi e^{i\pi/6}$$
$$I[(1+\cos\pi/3)+i\sin\pi/3]=\pi[\cos\pi/6+i\sin\pi/6]$$
$$R\cdot P:\ \ I[1+1/2]=\frac{\pi\sqrt{3}}{2}$$
$$\therefore I=\pi/\sqrt{3}$$
$$I\cdot P:\ I\left[\frac{\sqrt{3}}{2}\right]=\pi\cdot 1/2$$
$$I=\pi/\sqrt{3}$$

7. Evaluate $\int_0^\infty \frac{\log(1+x^2)}{x^{1+\alpha}}dx, 0<\alpha<1$

Solution:

Consider $f(z)=\frac{\log(1+z^2)}{z^{1+\alpha}}$

Singularities of $f(z)$ are given by,

$z^{1+\alpha}=0 \Rightarrow z=0$

Which lies on red axis.

$c=\Gamma\cup L_1\cup\gamma\cup L_2$

$f(z)$ is analytic inside and on C.

By Cauchy Theorem,

$$\int_C f(z)dz = 0$$

$$\int_\Gamma + \int_{L_1} + \int_2 + \int_{L_2} = 0.$$

$$\int_\Gamma \frac{\log(1+z^2)}{z^{1+\alpha}} dz + \int_{-R}^{-\gamma} \frac{\log(1+x^2)}{x^{1+\alpha}} dx - \int_{-\gamma}^{\gamma} \frac{\log(1+z^2)}{z^{1+\alpha}} dz + \int_{\gamma}^{R} \frac{\log(1+x^2)}{x^{1+\alpha}} dx = 0$$

…………… (1)

$$\lim_{z\to\infty} z\, f(z) = \lim_{z\to\infty} z \frac{\log(1+z^2)}{z^{1+\alpha}}$$

$$= \lim_{z\to\infty} z \frac{\log(1+z^2)}{z^{\alpha}}$$

$$= \lim_{z\to\infty} \frac{\log[z^2\left(\frac{1}{z^2}+1\right)]}{z^{\alpha}}$$

$$= \lim_{z\to\infty} \frac{\log[z^2] + log(1/z^2+1)}{z^{\alpha}}$$

$$= \lim_{z\to\infty} \frac{2\log z + log(1/z^2+1)}{z^{\alpha}}$$

$$= 2 \lim_{z\to\infty} \frac{1/z}{\alpha z^{\alpha-1}} + \lim_{z\to\infty} \frac{1}{z^{\alpha}} log(1/z^2+1)$$

$$= 2(0) + 0 = 0$$

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0 \qquad \text{…………. (2)}$$

$$\lim_{r\to 0} \int_\gamma \frac{\log(1+z^2)}{z^{1+\alpha}} dz = ik(\beta - \alpha)$$

$$= ik(\pi - 0)$$

$$= ik\pi \qquad \text{…………. (3)}$$

$$k = \operatorname*{Res}_{z=0} f(z)$$

$$= \lim_{z\to 0} (z-0) \frac{\log(1+z^2)}{z^{1+\alpha}}$$

$$= \lim_{z\to 0} \frac{\log(1+z^2)}{z^{\alpha}}$$

$$= \lim_{z\to 0} \frac{(z^2) - (z^2)^2/2 + (z^2)^3/3}{z^{\alpha}}$$

$$= 0\,, \ (\because 0 < \alpha < 1)$$

Sub in (3), $\lim_{r\to 0}\int_{\gamma}\frac{\log(1+z^2)}{z^{1+\alpha}}dz = 0$

Taking $\lim R\to\infty, r\to 0$ in (1) and using(2) &(4) we get,

$$0+\int_{-\infty}^{0}\frac{\log(1+x^2)}{x^{1+\alpha}}dx - 0 + \int_{0}^{\infty}\frac{\log(1+x^2)}{x^{1+\alpha}}dx = 0$$

put $x = -t \Rightarrow dx = -dt$

$$\int_{\infty}^{0}\frac{\log(1+(-t)^2)}{(-t)^{1+\alpha}}(-dt) + \int_{0}^{\infty}\frac{\log(1+x^2)}{x^{1+\alpha}}dx = 0$$

$$\int_{0}^{\infty}\frac{\log(1+t^2)}{(-1)^{1+\alpha}t^{1+\alpha}}dt + \int_{0}^{\infty}\frac{\log(1+x^2)}{x^{1+\alpha}}dx = 0$$

$$\frac{1}{(-1)(-1)^{\alpha}}I + I = 0\left[\because \frac{-1}{(e^{i\pi})\alpha}+1\neq 0,\ \left(e^{i\pi}\right)^{\alpha}\neq 1,^{0<\alpha<1}\right]$$

$$I\left[\frac{-1}{(e^{i\pi})^{\alpha}}+1\right] = 0 \Rightarrow I = 0$$

Type- II

1.Evaluate $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^3}dx,\ a$ real

Solution:

Consider the contour $C = \Gamma U L$.

Where $\Gamma: |z| = R$ (upper Semi-circle)

$L = [-R, R]$

$\int_{C=\Gamma\cup L}\frac{z^2}{(z^2+a^2)^3}dz = \int_{\Gamma} + \int_{L}$

$= \int_{\Gamma}\frac{z^2}{(z^2+a^2)^3}dz + \int_{-R}^{R}\frac{x^2}{(x^2+a^2)^3}dx$ ………… (1)

$\lim_{|z|\to\infty} zf(z) = \lim_{z\to\infty} z\cdot\frac{z^2}{(z^2+a^2)^3}$

$= 0.$

$$\Rightarrow \lim_{|z|\to\infty}\int_{\Gamma} f(z)dz = 0$$

Sub in (1).

Taking $\lim_{R\to\infty}$,

$$\lim_{R\to\infty}\int_{C}\frac{z^2}{(z^2+a^2)^3}dz = 0 + \int_{-\infty}^{\infty}\frac{x^2}{(x^2+a^2)^3}dx$$

$= 2\pi i(R_1 + R_2 + \cdots \ldots..)$ ………. (2)

poles are given by,

$(z^2 + a^2)^3 = 0$

$(z^2 + a^2) = 0$ thrice,

$z^2 = -a^2$ thrice,

$z = \pm ai$ thrice.

Among these poles,

$Z = ia$ alone lies inside c and it is a pole of order 3 . [take $a > 0$]

$$R_1 = \text{Res}_{z=a_i} f(z)$$

$$= \lim_{z\to ai} \frac{1}{2} D^{3-1}(z - ia)^3 \frac{z^2}{(z + ia)^3(z - ia)^3}$$

$$= \frac{1}{2} \lim_{z\to ai} D^2(z^2)(z + ia)^{-3}$$

$$= \frac{1}{2} \lim_{z\to ia} D[2z(z + ia)^{-3} + z^2(-3)(z + ia)^{-4}]$$

$$= \frac{1}{2} \lim_{z\to ia} 2[1(z + ia)^{-3} + z(-3)(z + ia)^{-4}] - 3[2z(z + ia)^{-4} + z^2(-4)(z + ia)^{-5}]$$

$$= \frac{1}{(2ia)^3} - \frac{3ia}{(2ia)^4} - \frac{3}{2}\left[\frac{2ia}{(2ia)^4} - \frac{4(ia)^2}{(2ia)^5}\right]$$

$$= \frac{1}{-8ia^3} - \frac{3}{16(-ia)^3} - \frac{3}{16}\left[\frac{1}{(-ia)^3}\right] + \frac{6}{32}\frac{1}{(-ia)^3}$$

$$= \frac{4 - 6 - 6 + 6}{-32ia^3} = \frac{-2}{-32ia^3}$$

$$\Rightarrow R_1 = \frac{1}{16ia^3}$$

Sub in (2),

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)^3} dx = 2\pi i \left[\frac{1}{16ia^3}\right]$$

$$= \frac{\pi}{8a^3}$$

2. Evaluate : $\int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}$

Solution:

Let $c = \Gamma U L$

Consider,

$$\int_C \frac{z^2 dz}{(z^2 + 9)(z^2 + 4)^2} = \int_\Gamma + \int_L$$

Taking $\lim_{k\to\infty}$

$$\lim_{R\to\infty}\int_c \frac{z^2dz}{(z^2+9)(z^2+4)^2} = 0 + \int_{-\infty}^{\infty}\frac{x^2dx}{(x^2+9)(x^2+4)^2}$$

$= 2\pi i(R_1 + R_2 + \cdots \ldots..)$ .... (1)

Poles are given by,

$$(z^2+9)(z^2+4)^2 = 0$$
$$z^2 = -9, z^2 = -4 \text{ twice}$$
$$z = \pm 3i,\ z = \pm 2i \text{ twice}$$

Among these poles.

$z = 3i$ (of order 1) and

$Z = 2i$ (of order 2) lies inside C.

$$R_1 = \text{Res}_{z=3i} f(z)$$
$$= \lim_{z\to 3i}(z-3i)\frac{z^2}{(z+3i)(z-3i)(z^2+4)^2}$$
$$= \frac{(3i)^2}{(6i)(-9+4)^2}$$
$$= \frac{-9}{6i(25)}$$
$$R_1 = \frac{-3}{50i}$$

$$R_2 = \text{Res}_{2=2i}\, f(\text{z})$$

$$= \lim_{z\to 2i} \frac{1}{2!} D^{2-1}\left[(z-2i)^2 \frac{z^2}{(z^2+9)[(2+2i)(z-2i)]^2}\right]$$

$$= \lim_{z\to 2i} \frac{1}{1} D\left[\frac{z^2}{(z^2+a)(z+2i)^2}\right]$$

$$= \lim_{z\to 2i} \frac{(z^2+a)(z+2i)^2(2z) - z^2[2z(z+2i)^2 + (z^2+9)2(2+2i)}{(z^2+a)^2(z+2i)^4}$$

$$= \frac{(-4+9)(-16)(4i) + 4[4i(-16) + (-4+9)2(4i)]}{(-4+a)^2(256)}$$

$$= \frac{-320i - 96i}{25(256)}$$

$$= \frac{-416i}{25\times 256}$$

$$= \frac{-13i}{200}$$

$$R_1 + R_2 = \frac{-3}{50i} - \frac{13i}{200}$$

$$= \frac{12i - 13i}{200}$$

$$= \frac{-i}{200}$$

Sub in (1)

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = 2\pi i(R_1 + R_2)$$

$$= 2\pi i(-i/200)$$

$$= \frac{2\pi}{200}$$

$$= \frac{\pi}{100}$$

$$\therefore \int_0^{\infty} \frac{x^2 dx}{(x^2+9)(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+9)(x^2+4)^2} dx$$

$$= \frac{1}{2}\left(\frac{\pi}{100}\right)$$

$$= \frac{\pi}{200}$$

Harmonic Functions:

4.2. Definition and Basic Properties:

Any real valued function $U(z)$ or $U(x,y)$ defined and single valued in a region Ω is said to be harmonic in Ω if it is continuous with its partial derivatives of first two orders and satisfies the laplace's equation,

$$\nabla u = \nabla^2 u(x,y) = 0$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

A Harmonic function is also called a potential function.

Note:

In polar co-ordinates the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ takes the form

$$r\frac{\partial}{\partial r}\left(r,\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0$$

Properties:

(1) Prove that the Sum of two harmonic function is a harmonic function.

Proof:

Let $u_1(x,y)$ and $u_2(x,y)$ be any two harmonic function in Ω.

Let $u(x, y) = u_1(x, y) + u_2(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2}{\partial x^2}(u_1 + u_2) + \frac{\partial^2}{\partial y^2}(u_1 + u_2)$$

$$= \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial y^2}$$

$$= 0 + 0 \qquad [\because u_1 \ \& \ u_2 \text{ are harmonic }]$$

$$= 0$$

$\therefore u = u_1 + u_2$ is harmonic.

(2) Prove that a constant multiple of a harmonic function is also harmonic function.

Proof:

Let u be any harmonic function in Ω and Let c be any constant.

Let $\phi = c\mu$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 (cu)}{\partial x^2} + \frac{\partial^2}{\partial y^2}(cu)$$

$$= c \cdot \frac{\partial^2 u}{\partial x^2} + c \cdot \frac{\partial^2 u}{\partial y^2}$$

$$= c\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2}\right)$$

$$= 0$$

$\therefore \phi = cu$ is harmonic.

(3) Prove that a simplest harmonic function is ax+by

Proof:

Let $u = ax + by$

$$\frac{\partial u}{\partial x} = a\,, \frac{\partial^2 u}{\partial x^2} = 0$$
$$\frac{\partial u}{\partial y} = b, \frac{\partial^2 u}{\partial y^2} = 0$$
$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0$$

is called harmonic function.

$\therefore u = ax + by$ is harmonic.

(4) Prove that a simplest harmonic function is $e^x \sin y$

$$u(x, y) = e^x \sin y$$
$$\frac{\partial u}{\partial x} = e^x \sin y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \sin y$$
$$\frac{\partial u}{\partial y} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$
$$\Delta e^{2x} \sin y - e^x \sin y = 0$$
$$\Delta u = \nabla^2 u = 0.$$

is called harmonic function.

(5) prove that a function log r is harmonic where $r >$ u.

Proof:

$$u(r, \theta) = \log r$$
$$\frac{\partial u}{\partial r} = \frac{1}{r}, \frac{\partial u}{\partial \theta} = 0, \Rightarrow \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$r \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\, \partial^2} = r \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{1}{r}\right) + 0$$
$$= r \cdot 0$$
$$= 0.$$

$\therefore \log r$ is harmonic.

(6) Prove that the function $a \log r + b$ is harmonic

Proof:

$$u(r,\theta) = a\log r + b$$
$$\frac{\partial u}{\partial r} = \frac{a}{r}$$
$$\frac{\partial u}{\partial \theta} = 0 \Rightarrow \frac{\partial^2 y}{\partial \theta^2} = 0$$
$$r \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = r \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{a}{r}\right) + 0$$
$$= r \cdot \frac{\partial}{\partial r}(a)$$
$$= r.0$$
$$= 0$$

$\therefore a\log r + b$ is harmonic.

(7) Prove that the argument θ is harmonic function.

Solution:

$$u(r,\theta) = \theta$$
$$\frac{\partial u}{\partial r} = 0, \frac{\partial u}{\partial \theta} = 1, \Rightarrow \frac{\partial^2 u}{\partial \theta^2} = 0$$
$$r \cdot \frac{\partial u}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{\partial^{2u}}{\partial \theta^2} = r \cdot \frac{\partial}{\partial r}(r,0) + 0$$

$= 0.$ $\quad \therefore \theta$ is harmonic.

Theorem 1:

If $u = u(x,y)$ is a harmonic in a region Ω. Prove that $f(z) = U_x - iU_y$ is analy tic in Ω.

Proof:

Let $f(z) = u + iv$

$f(x) = u_x - iu_y$ $\quad$ ……….(1) [$\because$ given]

$u = u_x, v = -u_y$ $\quad$ ………..(2)

clearly,

$u_x = u_{xx}$
$u_y = u_{xy}$
$v_x = -u_{yx}$

$u_x - v_y = u_{xx} + u_{yy}$
$= 0$

$u_x = v_y \quad \ldots\ldots\ldots..(4)$

$u_y + v_x = u_{xy} - u_{yx} = 0$

$\therefore u_y = -v_x \quad \ldots\ldots\ldots\ldots(5)$

from (4) & (5)

u and v satisfy Cauchy Riemann equation …………...(6)

From (3) & (6)

$f(z) = u + iv = u_x - iu_y$ is analytic in Ω

[Result:

$$f(z)dz = (u_x - iu_y)(dx + idy)$$
$$= (u_x dx + u_y dy) + i(-u_y dx + u_x dy)$$

R.P of $f(z)dz = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$

If u has a conjugate function v, then the imaginary part of $f(z)dz$.

(i.e.,) I.P of $f(z)dz = -u_y dx + u_x dy$

$= v_x dx + v_y dy$ [$\because V$ is conjugate harmonic of u]

$= dv \quad [\because u_x = v_y, u_y = -v_x]$

In general, however there is no single value conjugate function and in the circumstance it is better not to use the notation dv.

Instead,

$$* \, du = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

Called * du the conjugate differential of dv.

$f(z)dz = du + i * du$

$$\int_\gamma * \, du = \int_\gamma -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial y}dy = 0$$

for all cycles 2 which are homologous to zero in Ω.

Proof:

$$\int_\gamma f(z)dz = 0 \text{ [}\because f \text{ is analytic in } \Omega\text{].}$$

$$\int_\gamma du + i * du = 0$$

$$\int_\gamma du + i \int_\gamma * \, du = 0$$

$$0 + i \int_\gamma * \, du = 0[\because du = ex \text{ act diff and } v \text{ is closed].}$$

for all cycles 2 which are homolog us to zero in Ω

Theorem: 2

If u_1 and u_2 are harmonic in a region Ω then prove that $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ for every cycle γ which is homologous to zero in Ω.

Proof:

Given u_1 and u_2 are harmonic in Ω.

Let $* du$, and $* du_2$ be the harmonic conjugate differential of u_1 and u_2 respectively

$\therefore * du_1 = dv_1$ and $* du_2 = dv_2$ ....(1)

$$\begin{aligned} u_1 * du_2 - u_2 * du_1 &= u_1 dv_2 - u_2 dv_1 \\ &= u_1 dv_2 + (v_1 du_2 - v_1 du_2 - u_2 dv_1) \\ &= u_1 dv_2 + v_1 du_2 - (v_1 du_2 + u_2 dv_1) \\ u_1 * du_2 - u_2 * d\mu_1 &= u_1 dv_2 + v_1 du_2 - d(u_2 v_1) \quad \ldots\ldots\ldots\ldots..(2) \end{aligned}$$

Let $f_1(z) = u_1 + iv_1$

$$\begin{aligned} & f_2(z) = u_2 + iv_2 \\ & \therefore f_2'(z) = du_2 + idv_2 \\ & f_1(z) f_2'(z) = (u_1 + iv_1)(du_2 + idv_2) \\ & = (u_1 du_2 - v_1 dv_2) + i(u_1 dv_2 + v_1 du_2) \\ & \therefore \operatorname{Im}[f_1(z) f_2'(z)] = u_1 du_2 + v_1 du_2 \end{aligned}$$

Sub in (2), and Integrate over ν.

$$\begin{aligned} \int_\gamma u_1 * du_2 - u_2 * du_1 &= \int_\gamma \operatorname{Im} f_1(z) \cdot f_2'(z) dz - \int_\gamma d(u_2 v_1) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

Hence, $\int_\gamma \operatorname{Im} f_1(z)\, f_2'(z) = 0$ [by Cauchy's Theorem]

[$\because$ Im part of an analytic function is analytic] and $\int_\gamma d(u_2, v_1) = 0$

[$\because d(u_2, v_1)$ is exact diff].

Result:

If u is harmonic prove that $\int_\gamma * du = 0$

Proof:

If u_1 and u_2 are harmonic in Ω then $\int_\gamma u_1^* du_2 - u_2^* du_1 = 0$ [$\because$ above theorem]

Take $u_1 = 1$ and $u_2 = u =$ harmonic

$\therefore \int_\gamma 1 * du - u \times d(1) = 0 \Rightarrow$ (i.e.,) $\int_\gamma * du = 0$]

[Show in detail that the arithmetic mean of harmonic functions over concentric circles $|z| = r$ is a linear function of $\log r$ deduce mean Value properly of harmonic functions in a disc $|z| < R$]

4.3. The Mean-value property:

The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of $\log r$, $\frac{1}{2\pi}\int_{|z|=r} u d\theta = \alpha \log r + \beta$ and if u is harmonic in a disc, $\alpha = 0$ and the arithmetic mean is a constant.

Proof:

Take Ω as $0 < |z| < \rho$

Take $u_1 = \log r$ which is harmonic

Take $u_2 = u =$ harmonic function is Ω.

Take $\gamma = c_1 - c_2$

Where $c_1 \colon |z| = r_1$ and $c_2 \colon |z| = r_2$

where $r_1 < r_2$

Now, $\gamma = 0$ (mod Ω)

By the previous theorem,

$$\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$$

$$\int_{c_1 - c_2} \log r * du - u * d(\log r) = 0 \quad \ldots\ldots\ldots (1)$$

We know that (by result) on circle $|z| = r$

$$\left.\begin{aligned} * du &= r \cdot \frac{\partial u}{\partial r} d\theta \\ * d(\log r) &= r \cdot \frac{\partial}{\partial r} \log r d\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (2)$$

Sub (2) in (1),

$$\int_{c_1 - c_2} \log r \cdot r \frac{\partial u}{\partial r} d\theta - u \cdot r \frac{\partial}{\partial r} (\log r) d\theta = 0$$

$$\int_{c_1 (or) |z| = r_1} \log r \cdot r \cdot \frac{\partial u}{\partial r} d\theta - u \cdot r \frac{\partial}{\partial r} (\log r) d\theta$$

$$= \int_{c_2\,(or)|z| = r_2} \log r \cdot r \cdot \frac{\partial u}{\partial r} d\theta - ur \cdot \frac{\partial}{\partial r} (\log r) d\theta$$

$$\log r_1 \int_{c_1} r_1 \frac{\partial u}{\partial r} d\theta - \int_{c_1} u\gamma_1 \cdot \frac{1}{\gamma_1} d\theta$$

$$= \log r_2 \int_{C_2} r_2 \frac{\partial u}{\partial r} d\theta - \int_{c_2} u \cdot \gamma_2 \frac{1}{\gamma_2} d\theta \quad \text{............ (3)}$$

Hence, if u is a harmonic function in the annulus

$r_1 < |z| = r < r_2$
$r_1 < r < r_2$

From (3) we get,

$$\int_{|z|=r} u d\theta - \log r \int_{|z|=\gamma} r \cdot \frac{\partial u}{\partial r} d\theta = \text{ constant } = \beta'(\text{ say }) \quad \text{.......... (4)}$$

Since u is harmonic in Ω.

$$\int_{\gamma} * du = 0, \quad \forall \text{ cycle } r \equiv 0(\text{mod}\Omega) \quad [\because \text{ Result }]$$

$$\int_{c_1 - c_2} r \cdot \frac{\partial u}{\partial r} d\theta = 0$$

$$\int_{c_1} r \frac{\partial u}{\partial r} d\theta = \int_{c_2} r \cdot \frac{\partial u}{\partial r} d\theta$$

$$\Rightarrow \int_{|z|=r} r \frac{\partial u}{\partial r} d\theta = \text{constant} = \alpha' \quad (say) \quad \text{........... (5)}$$

Sub (5) in (4)

$$\int_{|z|=r} u d\theta - \log \gamma \cdot \alpha' = \beta'.$$

$$\int_{|z|=\gamma} u d\theta = \log r \cdot \alpha' + \beta'$$

divide by 2π, $\frac{1}{2\pi} \int_{|z|=\gamma} u d\theta = \frac{\alpha'}{2\pi} \log \gamma + \frac{\beta'}{2\pi}$

A. m of u over $|z| = \gamma = \alpha \log \gamma + \beta$ .... (6)

$=$ Linear f_n. of $\log r$.

II part:

To prove that : $\alpha = 0$ if u is harmonic throughout the disc $|z| \leq r$.

$$\alpha = \frac{\alpha'}{2\pi} = \frac{1}{2\pi}\int_{|z|=r} r\,\frac{\partial u}{\partial r}\cdot d\theta \quad [\because (5)]$$

$$= \frac{1}{2\pi}\int_{|z|=r} *\,du = \frac{1}{2\pi}\int_{|z|=r} dv \; [\because \; *\,du = dv]$$

$$= \frac{1}{2\pi}(0) \quad [\because dv \text{ is exact diff}\,]$$

$$\alpha = 0 \qquad \ldots\ldots\ldots\ldots..(7)$$

III Part:

If u is harmonic in Ω. Then prove that A.m of u over $|z| = r =$ constant.

Proof:

Sub (7) in (6)

$$\frac{1}{2\pi}\int_{|z|=\gamma} u d\theta = 0\cdot\log\gamma + \beta$$

$$= \beta$$

$$= \text{ constant}$$

(i.e.,) Arithmetic mean of u over $|z| = r = \beta =$ constant

Theorem 1:

Maximum Principle for Harmonic Function

A non-constant harmonic function has no maximum modulus in its region of definition consequently the maximum value on a closed bounded Set E is taken on the boundary of E.

(or)

In $u(z)$ is a non-constant harmonic function in the region Ω then prove that $|u(z)|$ has no maximum value in Ω.

Proof:

Part-I

Let $u(z)$ be a hon-constant harmonic function in the region Ω.

To Prove that: $\max|u(z)|$ is not obtained at any point in Ω.

Proof:

Suppose the theorem is false. then their exist $z_0 \in \Omega$ such that $|u(z_0)|$ is maximum ……(1)

consider $\gamma: |z - z_0| = r$ lying in Ω.

$\therefore z - z_0 = re^{i\theta}$, $0 \leq \theta \leq 2\pi$

$z = z_0 + re^{i\theta}$ is any point on γ

By mean value property of harmonic function $u(z)$.

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

$|u(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\, u(z_0 + re^{i\theta})|\,|d\theta|$ ………….. (2)

From (1), $|u(z_0 + re^{i\theta})| \leqslant |u(z_0)|$ …………(3)

To prove that $|u(z_0 + re^{i\theta})| = |u(z_0)|$

Suppose the inequality (<) in (3) is true for a single value of θ.

⇒ By continuity,

The inequality in (3) is true on an arc

(i.e.,) $|u(z_0 + re^{i\theta})| < |u(z_0)|$ on an arc of γ …………(4)

From (2)

$|u(z_0)| \leqslant A.$M of $|\, u(z_0 + re^{(\theta)}\,|$ on γ ………… (5)

From (4)

A.M of $|u(z_0 + re^{io})| < |u(z_0)|$ …………….(6)

Sub (5) in (6)

$|u(z_0)| < |u(z_0)|$

This contradiction prove that our assumption

$|u(z_0 + re^{i\theta})| < |u(z_0)|$ in γ.

This is true for all $\gamma \geq 0$.

∴ $|u(z)|$ is a constant and equal to $|u(z_0)|$ is a nod of z_0.

∴ $u(z)$ is a constant in a neighbourhood of z_0 and hence in Ω.

This contradiction proves that max$|u(z)|$ is not attained at any point z_0 in Ω.

Part - II

Let E be a closed bounded set

Since $U(z)$ is harmonic, it is continuous and hence maximum modulus $u(z)$ is taken at Some point of E.

By, Max$|u(z)|$ is not attained at any interior point of E

∴ Max $|u(z)|$ is attained at some point on the boundary of E :

Theorem 2: (Minimum Principle for Harmonic Function)

A non-constant harmonic function has no minimum modulus in its region of definition consequently the minimum on a closed set E is taken an the boundary of E.

Proof:

Let $u(z)$ be a harmonic function in a region Ω then,

$V(z) = -u(z)$ is harmonic in Ω

Applying Max. principle to $v(z) = -u(z)$.

we get, Minimum principle for $u(z)$

4.4. Poisson's Formula:

Theorem: 1 (Poisson's Formula for Harmonic Function)

If (i) $u(z)$ is harmonic for $|z| < R$ and

(ii) $u(z)$ is continuous for $|z| \leq R$

then, $u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2} u(z) d\theta, \quad \forall |a| < R.$

Proof:

Let $u(z)$ be Harmonic for $|z| \leq R$.

we know that,

the linear transformation $Z = S(\zeta) = \frac{R(R\zeta+a)}{R+\bar{a}\zeta}$ ....(1)

Maps $|\zeta| \leq 1$ onto $|z| \leq R$

when $\zeta = 0$,

$$z = \frac{R(0+a)}{R} = \frac{Ra}{R} = a \Rightarrow z = a$$

$\zeta = 0$ corresponds to $z =$ a ... ... (2)

Since $u(z)$ is harmonic for $|z| \leq R$.

$u(S(\zeta))$ is harmonic in $|\zeta| \leq 1$ [$\because$ (1)]

We know that,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z + re^{i\theta}\right) d\theta$$

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(z) d\theta \left[\because z_0 = re^{i\theta}, z_0 = 0 \Rightarrow \mathrm{r} = 0\right]$$

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u\big(s(\zeta)\big) d(\arg(\zeta)) \quad \ldots\ldots\ldots\ldots (4)$$

From (1),

$$z(R + \bar{a}\zeta) = R(R\zeta + a)$$
$$zR - Ra = R^2\zeta - z\bar{a}\zeta$$
$$R(z - a) = \zeta(R^2 - \bar{a}z)$$
$$\zeta = \frac{R(z-a)}{(R^2 - \bar{a}z)} \quad \ldots\ldots\ldots\ldots(5)$$

in the inverse transformation of (1)

Now,

$$|\zeta| = 1 \Rightarrow \zeta = 1 \cdot e^{i\phi}$$
$$\Rightarrow \log\zeta = i\phi\log e = i\phi.1$$

Taking differentials

$$\frac{1}{\zeta}d\zeta = id\phi$$
$$d\phi = \frac{1}{i}\frac{d\zeta}{\zeta}$$
$$d(\arg\varphi) = d\phi = -i\frac{d\zeta}{\zeta} \quad \ldots\ldots\ldots.(6)$$

taking log in (5),

$$\log\zeta = \log(R(z-a)) - \log(R^2 - \bar{a}z)$$
$$= \log R + \log(z-a) - \log(R^2 - \bar{a}z)$$

Taking differentials,

$$\frac{1}{\zeta}d\zeta = 0 + \frac{1}{z-a}dz - \frac{1}{R^2-\bar{a}z}(-\bar{a})dz$$
$$\frac{d\zeta}{\varphi} = \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right)dz \ldots\ldots\ldots\ldots(7)$$

Now, $|z| = R \Rightarrow z = Re^{i\theta}$

$$dz = Re^{i\theta}id\theta$$
$$dz = zid\theta \quad \ldots\ldots\ldots\ldots..(8)$$

Sub (8) in (7)

$$\frac{d\zeta}{\zeta} = \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2-\bar{a}z}\right)izd\theta \quad \ldots\ldots\ldots\ldots\ldots(9)$$

Sub (9) in (6)

$$d(\arg\zeta) = -i\left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z}\right)izd\theta$$
$$= \left(\frac{z}{z-a} + \frac{z\bar{a}}{R^2 - \bar{a}z}\right)d\theta$$
$$= \left(\frac{z}{z-a} + \frac{z\bar{a}}{z\bar{z} - \bar{a}z}\right)d\theta \quad [\because R^2 = |z|^2 = z\bar{z}]$$

$= \left(\frac{z}{z-a} + \frac{\bar{a}}{\bar{z}-\bar{a}}\right) d\theta$ …………… (10)

Sub (10) in (4) $u(a) = \frac{1}{2\pi}\int_{|=|=1} u(S(\zeta))\frac{R^2-|a|^2}{|z-a|^2} d\theta$ ………………(11)

Corollary:1 (**Another form of Poisson Formula**)

Prove that $u(a) = \frac{1}{2\pi}\int_{|z|=R} Re\left(\frac{z+a}{z-a}\right)u(z)d\theta$.

Proof:

$$\text{Re}\left(\frac{z+a}{z-a}\right) = \frac{1}{2}\left[\frac{z+a}{z-a} + \left(\overline{\frac{z+a}{z-a}}\right)\right] \qquad \left[\because \text{Re}\, z = \frac{z+\bar{z}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{z+a}{z-a} + \frac{\bar{z}+\bar{a}}{\bar{z}+\bar{a}}\right]$$

$$= \frac{1}{2}\left[\frac{(z+a)(\bar{z}-\bar{a}) + (z-a)(\bar{z}+\bar{a})}{(z-a)(\bar{z}-\bar{a})}\right]$$

$$= \frac{1}{2}\left[\frac{z\bar{z} - z\bar{a} + a\bar{z} - a\bar{a} + z\bar{z} + z\bar{a} - a\bar{z} - a\bar{a}}{(z-a)(\bar{z}-\bar{a})}\right]$$

$$= \frac{1}{2}\left[\frac{|z|^2 - |a|^2 + |z|^2 - |a|^2}{|z-a|^2}\right]$$

$$= \frac{1}{2}\, 2\left[\frac{R^2 - |a|^2}{|z-a|^2}\right] \quad \text{………….} (12)$$

Sub (12) in (11)

$$u(a) = \frac{1}{2\pi}\int_{|z|=R} Re\left(\frac{z+a}{z-a}\right)u(z)d\theta \quad \text{…………}(13)$$

Corollary:2

Poisson Formula for Harmonic function in Polar Co-ordinates

Prove that $u(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2} u(Re^{i\theta})d\theta$

Proof:

In equation (11) [previous Theorem]

circle $|z| = R \Rightarrow z = Re^{i\theta}, 0 \le \theta \le 2\pi$

Let $a = re^{i\phi} \Rightarrow |a| = r$.

$$\frac{R^2-|a|^2}{|z-a|^2}=\frac{R^2-r^2}{(z-a)(\bar{z}-\bar{a})}\ [\because |a|^2=r^2]$$
$$=\frac{R^2-r^2}{(Re^{i\theta}-re^{i\phi})(Re^{i\theta}-re^{-i\phi})}$$
$$=\frac{R^2-r^2}{R^2-Re^{i\theta}re^{-i\phi}-re^{i\phi}Re^{-i\theta}+r^2}$$
$$=\frac{R^2-r^2}{R^2-rR[e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}]+r^2}$$
$$=\frac{R^2-r^2}{R^2-rR\,2\cos(\theta-\phi)+r^2}\quad\ldots\ldots\ldots\ldots\ldots(14)$$

Sub (14) and $z=Re^{i\theta}$ in (11)

$$u(r,e^{i\phi})=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\phi)+r^2}u(Re^{i\theta})d\theta.\quad\ldots\ldots\ldots\ldots(15)$$

Note:

(11), (13), (15) are three different forms of Poisson formula for Harmonic functions

Corollary:3 (Derive Schwarz's Formula)

We know that, the another form of Poisson formula.

$$u(a)=\frac{1}{2\pi}\int_{|z|=R}\text{Re}\left(\frac{z+a}{z-a}\right)u(z)d\theta$$
$$u(a)=\text{Re}\left(\frac{1}{2\pi}\int_{|z|=R}\left(\frac{z+a}{z-a}\right)u(z)d\theta\right)$$

Consider $|\zeta|=1$

$$\zeta=e^{i\theta}$$
$$d\zeta=e^{i\theta}\cdot id\theta$$
$$\frac{d\zeta}{\zeta i}=d\theta$$

Changing a by z and z by ζ,

$$u(z)=\text{Re}\left(\frac{1}{2\pi}\int_{|\zeta|=R}\left(\frac{\zeta+z}{\zeta-z}\right)u(\zeta)d\theta\right)$$
$$u(z)=\text{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=R}\left(\frac{\zeta+z}{\zeta-z}\right)u(\zeta)d\theta\right)$$

The expression in the bracket in an analytic function of z.

If follows that $u(z)$ in the real part of $f(z)$, Where $u(z) = \frac{1}{2\pi i}\int_{|\zeta|=R}\left(\frac{\zeta+z}{\zeta-z}\right)u(\zeta)d\theta\frac{d\zeta}{\zeta} + ic$

[Note that $u(z) = 0$, and $u(z)$ is harmonic then if possible to find $v(z)$ harmonic function. Such that $f(z) = u(z) + iv(z)$ is analytic]

Where C is an arbitrary real constant.

Theorem 2: (Maximum Principle on Harmonic Function)

A non-constant harmonic function has no maximum modulus in the region of definition

Consequently the maximum value on a closed bounded set E is taken on boundary of E.

Proof:

Part - I

Let $u(z)$ is non-constant harmonic function in the region Ω

To prove that Max $|u(z)|$ is not obtained at any in Ω

Since $u(z)$ in continuous, |u(z)| is also continuous in the given closed and bounded. region Ω and hence attains its bounds.

(i.e.,) $|u(z)| \leq M$, for all $z \in \Omega$ and $M > 0$ ………..(1)

If possible,

Let $|u(z)|$ attain its maximum value M at some interior point, $z_0 \in \Omega$.

(i.e.,) $|u(z_0)| = m$ …………(2)

Construct a circular disk $|z - z_0| \leq r$ contained $in\ \Omega$

By Mean value property,

we have,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta$$

$$|u(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} |u(z_0 + re^{i\theta})|d\theta$$

$$\therefore |u(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |u(z_0 + re^{i\theta})|d\theta$$

From (1) and (2),

we have $|u(z)| \leq |u(z_0)|$ on the circular disc $|z - z_0| = \gamma$.

we have $z = z_0 + re^{i\theta}$

$\left|u\left(z_0 + re^{i\theta}\right)\right| \leq |u(z_0)|$ ………….. (4)

from (3) & (4), we get

$$|u(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|u\left(z_0 + re^{i\theta}\right)\right| d\theta$$

(i.e.) $2\pi|u(z_0)| = \int_0^{2\pi} \left|u\left(z_0 + re^{i\theta}\right)\right| d\theta$

(i.e.) $\int_0^{2\pi} |u(z_0)| d\theta = \int_0^{2\pi} \left|u\left(z_0 + re^{i\theta}\right)\right| d\theta$

(i.e.) $\int_0^{2\pi} \left[|u(z_0)| - \left|u\left(z_0 + re^{i\theta}\right)\right|\right] d\theta = 0$

Since the integral is continuous and non-negative,

we obtain, $|u(z_0)| - \left|u\left(z_0 + re^{i\theta}\right)\right| = 0$

(i.e.) $\left|u\left(z_0 + re^{i\theta}\right)\right| = |u(z_0)|$

(i.e.) $|u(z)| = |u(z_0)|$ for z on the circular disc $|z - z_0| = r$

∴ By continuity,

$|u(z)| =$ constant

(i.e.) $u(z) =$ constant.

This is contradiction.

∴ $|u(z)|$ cannot attain its maximum in the interior of Ω

(i.e.) $|u(z)|$ attains its maximum value only on the boundary.

UNIT – V

Harmonic Functions and Power Series Expansions:

Schwarz Theorem - The reflection principle Weierstrass's Theorem – The Taylor's Series – The Laurent series.

Chapter 5: Section 5.1 - 5.5

5.1. Schwarz's Theorem

Theorem serves to express a given harmonic function through its values in a circle.

But the R.H.S of formula.

$$u(a) = 1/2\pi \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z) d\theta$$

has a meaning as soon as u is defined in $|z| = R$.

The equation is, does it have the boundary value $u(z)$ on $|z| = R$

Now choosing $R = 1 \ni u \to P_u$ in $a + vc$

Now we define Poisson integral of $u[P_u(z)]$.

Poisson Integral of $U(P_u(z))$

Let $U(\theta)$ be piecewise continuous function in $0 \leq \theta \leq 2\pi, |z| = R = 1$, we define

$P_U(z) = 1/2\pi \int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) U(\theta) d\theta$ is called Poisson integral of U]

Note:

(1) we observe that $P_U(z)$ is not only a function of z but also a function of the function U;

$\therefore P_u(z)$ is called functional

(2) The functional is linear functional,

For $P_u + \text{v} = P_u + P_V$

$P_{CU} = CP_U$, for constant C_i.

(3) Moreover $u \geq 0 \Rightarrow P_u(z) \geq 0$;

Because of this property P_U is said to be positive linear functional]

(4) Now $P_U = c$ where c is constant from this property

together with the linear and positive character of the functional, it follows that any inequality

$m \leq U \leq M \Rightarrow m \leq P_U \leq M$

Theorem: 1 (Schwarz Theorem)

The function $P_U(z)$ is harmonic for $|z| < 1$ and $\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$ provided that U is colts at θ_0

Proof:

First, we have to

Prove that $P_U(z)$ is harmonic

$$P_u(z) = 1/2\pi \int_0^{2\pi} \text{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) U(\theta) d\theta$$
$$= \text{Re} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right) \cup (\theta) d\theta$$

The expression in the product in Analytic function for $|z| < 1$

We know that,

The real and imaginary part of the analytic function is harmonic.

If follows that, $P_U(z)$ is a real part of the Analytic function.

$\therefore P_U(z)$ is harmonic for $|z| < 1$

Now we have to prove,

$\lim_{z \to e^{i\theta_0}} P_U(z) = U(\theta_0)$ without loss of generality.

We may assume that

$U(\theta_0) = 0$

Now Prove that $\lim_{z \to e^{i\theta_0}} P_U(z) = 0$

(i.e.,) we have prove $|P_U(z)| < \varepsilon$, whenever $\left|z - e^{i\theta_0}\right| < \delta$

For this,

Let c_1 and c_2 be 2 complementary ares of the unit circle $|z| = 1$

Let us choose $|u(\theta)| < \varepsilon/2$ …………(1) if $e^{i\theta} \in C_2$

Now we define

$$U_1(\theta) = \begin{cases} U(\theta) & \text{if } e^{i\theta} \in c_1 \\ 0 & \text{if } e^{i\theta} \in c_2 \end{cases}$$
$$U_2(\theta) = \begin{cases} U(\theta) & \text{if } e^{i\theta} \in c_2 \\ 0 & \text{if } e^{i\theta} \in c_1 \end{cases}$$

Then $u(\theta) = u_1(\theta) + u_2(\theta), \ \forall\theta,$

$P_u = P_{u_1} + P_{u_2} = Pu_1 + u_2$

Thus $P_u = P_{u_1} + P_{u_2}$ ………….(2).

Now,

$$\text{Re}\left(\frac{e^{i\theta}+z}{e^{i0}-z}\right) = \text{Re}\left[\frac{(e^{i\theta}+z)(e^{-i\theta}-\bar{z})}{(e^{i\theta}-z)(e^{-i\theta}-\bar{z})}\right]$$

$$= \text{Re}\left[\frac{1-\text{z}\bar{e}^{i\theta}+ze^{-\text{i}\theta}-z\bar{z}}{(e^{i\theta}-z)^2}\right]$$

$$= \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

which vanishes $|z| = 1$ and $e^{i\theta} \neq z$

Now,

$$P_{U_1}(z) = 1/2\pi \int_0^{2\pi} \text{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) U_1(\theta)d\theta$$

$\therefore P_{U_1}(z) = 0$ if $z = e^{i\theta} \in C_2$

(i.e.,) It is continuous.

$P_{U_1}(\text{z}) \longrightarrow 0$ as $z \longrightarrow e^{i\theta} \in c_2$

(i.e.,) $\lim_{z\to e^{i\theta_0}} P_{u_1}(z) = 0$

$\Rightarrow |P_{u_1}(z)| < \frac{\varepsilon}{2}$, for $|z - e^{i\theta}| < \delta$ ………… (3)

By (1),

$|u(\theta)| < \varepsilon/2$, if $e^{i\theta} \in C_2$

$\therefore |U_2(\theta)| < \varepsilon/2$, if $e^{i\theta} \in c_2$ and $|z| < 1$.

$\therefore$ Since, $|P_{u_2}(z)| < \varepsilon/2$ if $|z| < 1$ …………..(4)

From (2), we have $P_u(z) = P_{u_1}(z) + P_{u_2}(z)$

$$|P_u(z)| \leqslant |P_{u_1}(z)| + |P_{u_2}(z)| \quad [\because (3) \,\&\, (4)]$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for } |z - e^{i\theta}| < \delta \;\; |z| < 1$$

$$\Rightarrow |P_u(z)| < \varepsilon \text{ whenever } |z - e^{i\theta_0}| < \delta \; |z| = 1$$

$$\Rightarrow \lim_{z\to e^{i\theta_0}} P_u(z) = 0.$$

5.2. The Reflection principle:

Theorem 1:

Let Ω^+be the part in the upper half plane of a symmetric region Ω and let σ be the part of the real axis in Ω. Suppose that $V(x)$ is continuous in $\Omega^+ \cup \sigma$, harmonic in $\Omega^+$and zero on σ. then v has a harmonic extension to Ω.which satisfies the symmetry relation $v(\bar{z}) = -v(z)$

In the same situation, if v is the imaginary part of an analytic fun $f(z)$ in Ω^+then $f(z)$ has an analytic extension which satisfies $f(z) = \overline{f(\bar{z})}$

Proof:

Let us construct the fun $V(z)$ which is equal to $v(z)$ in Ω^+. o on σ, and equal to $-v(\bar{z})$ in the minor image of Ω^+.

we have to show that v is harmonic on σ for a point $x_0 \in \sigma$ consider a disc with center x_0 contained in Ω.

Let P_v denote the poisson integral with respect to this disc formed with the boundary values V.

The difference V - P_V is harmonic in the upper half of the disc. It vanishes on the half circle (by Schwarz theorem) and also on the diameter, because V tends to zero by deft and vanishes by symmetry.

The maximum and minimum principle implies $V = P_v$ in the up on half disc and the same proof can be repeated for the lower half.

we conclude that v is harmonic in the whole disc, and is particular at x_0.

For the remaining part of the theorem,

Let us consider once again a disc with center on σ.

We have already extended v to the whole disc, and v has a conjugate harmonic Function $-u_0$ in the same disc which we may Aarmolize.

So that $u_0 = \text{Re}\, f(z)$ in the upper half consider,

$v_0(z) = u_0(z) - u_0(\bar{z})$

$\frac{\partial v_0}{\partial x} = 0$ on the real diameter.

Also $\frac{\partial v_0}{\partial y} = 2\frac{\partial u_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0$

$\therefore$ The analytic function $\frac{\partial v_0}{\partial x} - i\frac{\partial v_0}{\partial y}$ vanishes on the real axis and hence identically

$\therefore u_0$ is a constant and this constant is clearly zero we have proved that $u_0(z) = u_0(\bar{z})$.

The construction can be repeated for arbitrary disc u_0 concedes in the over lapping discs

The definition can be extended to all of Ω and the the follows.

Example 1:

If $f(z)$ is analytic in the whole plane and real axis, purely imaginary on the imaginary axis, show that $f(z)$ is odd.

Solution:

By hypothesis.

Ω is symmetrical about the real and imaginary axis.

$\therefore f(z) = f(\bar{z})$ and $f(z) = -\overline{f(-\bar{z})}$

$\therefore \overline{f(\bar{z})} = -\overline{f(-z)}$

$f(\bar{z}) = -f(-\bar{z})\ \forall \bar{z}$

(i.e.,) $f(z) = -f(-z)$

$\therefore f$(z) is odd.

Example 2:

If $f(z)$ is analytic in $|z| \leqslant 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

Solution:

By the maximum principle $|f(z)| < 1$ for $|z| < 1$

The inverse of $|z| < 1$ is $|z| > 1$ & hence by reflection principle $f(z)$ has an analytic continuation to the whole extended plane

Show that if z and z^* are conjugate with respect to $|z| = 1$, then so are $f(z)$ and $f(z^*)$

Now $f(z)$ can have only a finite no of zeros in $|z| \leq 1$

Moreover if can have no zero in $|z| > 1$

For if $f(z) = 0$ in $|z| > 1$ then $f(z^*) = \infty$ & $|z^*| = 1$ while $f(z)$ is analytic in $|z| < 1$.

Also if $|z_0| > 1$ and z_0 is a pole of $f(z)$, then $f(z_0) = \infty$, which implies $f(z_0^*) = 0$.

(i.e.,) z_0^* is a zero of $f(z)$ & $|z^*| < 1$

which means that $f(z)$ can have only a finite number of poles in $|z| < 1$.

Thus $f(z)$ has only a finite no. of poles in the extended plane.

Hence $f(z)$ is rational.

Power Series Expansions:

5.3. Weierstrass's Theorem:

Theorem 1:

Suppose that $f_n(z)$ is analytic in the in the region Ω_n, and that the seq $\{f_n(z)\}$ converges to a limit fun $f(z)$ in a region Ω, uniformly on every Compact subset of Ω.Then $f(z)$ is analytic in Ω. Moreover $f_n'(z)$, converges uniformly to $f'(z)$ on every compact Subset of Ω.

Proof:

Let $\Omega = \Omega_1 \cup \Omega_2 \cup ....$

Then proof based on integral formula.

$$f_n(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\zeta)}{(\zeta - z)} d\zeta$$

where c is a circle $|\zeta - a| = r$

& $|z - a| < r(|z - a| = \rho)$

Taking limit on both sides,

$$\lim_{n\to\infty} f_n(z) = \frac{1}{2\pi i}\int_c \frac{\lim_{n\to\infty} f_n(\zeta) d\zeta}{(\zeta - z)}$$

Since $f_n(z)$ is uniformly cogs on every compact subset Ω of $f(z)$.

∴ we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta) d\zeta}{(\zeta - \mathrm{z})}$$

Since ζ is a arbitrary point.

∴ By apply the cauchy Integral formula, ∴ $f(z)$ is analytic in Ω

To prove : $f_n'(z)$ convergent to $f'(z)$ on every compact subset of Ω.

Now we consider,

$$f_n'(z) = \frac{1}{2\pi i}\int_c \frac{f_n(\zeta) d\zeta}{(\zeta - z)^2}$$

Also,

$$f'(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta) d\zeta}{(\zeta - z)^2}$$

$$\therefore f_n'(z) - f'(z) = \frac{1}{2\pi i}\int_c \frac{[f_n(\zeta) - f(\zeta)]}{(\zeta - z)^2} d\zeta$$

Taking modulus on both sides

$|f_n'(z) - f'(z)| < \frac{1}{2\pi}\int_C \frac{|f_n(\zeta)-f(\zeta)|}{|\zeta - z|^2}|d\zeta|$ ………… (1)

Since f_n(z) convergences uniformly to $f(z)$ on every compact sub set of Ω

we have to,

Every $\varepsilon > 0$, their exist a positive integer n_0

$|f_n(x) - f(x)| < \varepsilon, \quad n > 0, \quad \forall z \in C$

Now, $|\zeta - z| = |\zeta - a + a - z|$

$\geq |\zeta - a| - |z - a|$

$\geq \gamma - \rho$

$\therefore \frac{1}{|\zeta - z|} \leqslant \frac{1}{r-p}$ and $\frac{1}{|\zeta - z|^2} \leqslant \frac{1}{(\gamma-\rho)^2}$

$\therefore (1) \Rightarrow$

$$|f_n'(z) - f'(z)| \leq \frac{1}{2\pi}\frac{\varepsilon}{(r-\rho)^2}\int_C |d\zeta|$$

$$\leq \frac{1}{2\pi}\frac{\varepsilon}{(r-p)^2}2\pi r$$

$$\leq \frac{\varepsilon}{r^2(1-\rho/r)^2}r$$

$$\leq \frac{\varepsilon}{\gamma(1-\rho/r)^2}$$

$\rightarrow \sigma$ as $\gamma \rightarrow \infty$

$\therefore f_n'(z)$ convergences uniformly to $f'(z)$ where

$|z - a| < \rho \leq r$.

Since any compact subset of Ω can be covered by a finite no.of closed disc

Convergence uniformly on every compact subset.

5.4. Taylor's Series:

Theorem 1:

If $f(z)$ is analytic in a region Ω containing z_0 then the representation.

$$f(z) = f(z_0) + \frac{f'(z_0)(z - z_0)}{1!} + \cdots \ldots + \frac{f^n(z_0)(z - z_0)^n}{n!}$$

is valid in the largest open disc of center z_0 contained in Ω.

Proof:

Let c be the circle $|z - z_0 \ | = \rho$

S.T the closed disk $|z - z_0| \leq \rho$ contained in Ω Since $f(z)$ is analytic in the region Ω containing z. We can write, By Taylor's theorem,

$$f(z) = f(z_0) + \frac{f'(z)(z-z_0)}{1!} + \frac{f''(z_0)(z-z_0)^2}{2!} + \cdots\cdots + \frac{f^n(z_0)(z-z_0)^n}{n!} + f_{\mathrm{n+1}}(z)(z-z_0)^{\mathrm{n+1}}$$

……… (1)

$f_{n+1}(z) = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-z_0)^{n+1}(t-z)}$ ……….. (2)

Where ' c ' is the circle

$|t - z_0| = \rho$ …………. (3)

Their exist the closed disk $\| -z_0 \, | \leq P$ contained in Ω.

Let $\max|f(z)| = \mu$ on c. (or)

Max $|f(t)| = \mu$ on c ………..(4)

from (2)

$$|f_{n+1}(z)| = \left|\frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-z)(t-z_0)^{n+1}}\right|$$

$$\leq \frac{1}{2\pi}\int_c \frac{|f(t)||dt|}{|t-z||t-z_0|^{n+1}}$$

Let μ denote the max on $|f(t)|$ on c

$$|f_{n+1}(z)| \leq \frac{1}{2\pi}\int_c \frac{\mu}{\rho^{n+1}|t-z|}|dt| \quad \ldots\ldots\ldots\ldots (5)$$

$$\begin{aligned}[|t-z| &= |t - z_0 + z_0 - z| \\ &\geq |t-z_0| - |z-z_0| \\ &\geq \rho - |z-z_0|\end{aligned}$$

$$\therefore \frac{1}{|t-z|} \leq \frac{1}{\rho - |z-z_0|}\Bigg]$$

$$|f_{n+1}(z)| \leq \frac{1}{2\pi}\int_c \frac{\mu}{\rho^{n+1}(\rho - | z - z_0 1)}|dt|$$

$$\leq \frac{1}{2\pi}\frac{\mu}{\rho^{n+1}(\rho - |z-z_0|)}\int_c |dt|$$

$$\leqslant \frac{1}{2\pi}\int \frac{\mu}{p^{n+1}(\rho - |z-z_0|)}\cdot 2\pi\rho$$

$$\leq \frac{1}{2\pi}\frac{\mu}{\rho^{n+1}}\frac{1}{\rho\left(1-\frac{|z-z_0|}{\rho}\right)}2\pi\rho$$

$$|f_{n+1}(z)| \leqslant \frac{\mu}{\rho^{n+1}\left(1-\frac{1z-z_0 1}{\rho}\right)}$$

multiply by $\Rightarrow (z-z_0)^{n+1}$

$$\Rightarrow |(z-z_0)^{n+1}f_{n+1}(z)| \leq \frac{|z-z_0|^{n+1}}{\rho^{n+1}}\cdot\frac{\mu}{\left(1-\frac{|z-z_0|}{\rho}\right)}$$

$$|(z-z_0)^{n+1}f_{n+1}(z)| \leqslant \left[\frac{(z-z_0)}{\rho}\right]^{n+1}\frac{\mu}{\left(1-\frac{|z-z_0|}{\rho}\right)}$$

[Now consider,

open disk a disk $|z-z_0| \leq r \leq \rho$

$$\Rightarrow \frac{|z-z_0|}{\rho} < 1 \Rightarrow \left[\frac{|z-z_0|}{\rho}\right]^{n+1} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow |f(z)(z-z_0)^{n+1}| \to 0 \text{ as } n \to \infty$$

Since $|z-z_0| < \rho \longrightarrow 0$ as $n \longrightarrow \infty$.

$$0 \leq \frac{|z-z_0|}{\rho} < 1\Big]$$

∴ Sub in (1) we get,

$$f(z) = f(z_0) + \frac{f'(z_0)(z-z_0)}{1!} + \frac{f''(z_0)(z-z_0)^2}{2!} + \frac{f^n(z_0)(z-z_0)^n}{n!} + \cdots \ldots ....$$

This is known as Taylor's Series.

5.5. Laurent Series:

Theorem 1:

If $f(z)$ is defined analytic in the annulus region $R_1 < |z-a| < R_2$. Then $f(z)$ can be written in the form $f(z) = \sum_{n=-\infty}^{\infty} A_n(z-a)^n$

Where $A_n = \frac{1}{2\pi i}\int_{|\zeta-a|=\gamma}\frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}}$

Proof:

Consider the circle having the center at $z = a$ and radius γ their exist $R_1 < \gamma < R_2$

Define $f_1(z)$ and $f_2(z)$ as follows,

$$f_1(z) = \frac{1}{2\pi i}\int_{c\ (or)|\zeta-a|=\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \ \ ....(1) \text{ for } |z-a| < r < R_2$$

$$f_2(z) = \frac{-1}{2\pi i}\int_{|\zeta-a|=\gamma} \frac{f(\zeta)d\zeta}{\zeta-z} \ \ ... ...(2) \ \text{ for } R_1 < r < |z-a| \text{ clearly, } R_1 < r < R_2$$

By Cauchy Theorem,

The value of the integral (1) and (2) do not change $w.r.$ to ' r '

$\therefore$ $f_1(z)$ & $f_2(z)$ are uniquely define they represent analytic in $|z-a| < R_2$ and

$|z-a| > R_1$ respectively.

By Cauchy integral Theorem,

$f(z) = f_1(z) + f_2(z)$

Which is analytic in the annulus $R_1 < |z-a| < R_2$

Now we consider, the Taylor is development of F_1:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a + a - z}$$

$$= \frac{1}{(\zeta - a) - (z - a)}$$

$$= \frac{1}{(\zeta - a)\left[1 - \frac{(z-a)}{\zeta - a)}\right]}$$

$$= \frac{1}{(\zeta - a)}\left(1 - \left(\frac{z-a}{g-a}\right)\right)^{-1}$$

$$= \frac{1}{(\zeta - a)}\left[1 + \frac{z-a}{\zeta - a} + \left(\frac{z-a}{\zeta - a}\right)^2 + \cdots ... . \right] \text{ if } \left|\frac{z-a}{\zeta - a}\right| < 1$$

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a} + \left(\frac{(z-a)}{(\zeta-a)^2}\right) + \frac{(z-a)^2}{(\zeta-a)^3} + ...$$

$$\therefore f_1(z) = \frac{1}{2\pi i}\int_{|\zeta-a|=\gamma} \frac{f(\zeta)d\zeta}{(\zeta - z)}$$

$$= \frac{1}{2\pi i}\int_{|\zeta-a|=\gamma} f(\zeta)\left[\frac{1}{(\zeta-a)} + \frac{(z-a)}{(\zeta-a)^2} + \cdots\right] d\zeta$$

$$= \frac{1}{2\pi i}\int_{|\zeta-a|=\gamma} \frac{f(\zeta)d\zeta}{(\zeta-a)} + \left(\frac{z-a}{2\pi i}\right)\int_{|\zeta-a|=r} \frac{f(\zeta)d\zeta}{(\zeta-a)^2} + \cdots$$

$$f_1(z) = \sum_{n=0}^{\infty} A_n (z-a)^n$$

where $A_n = \frac{1}{2\pi i} \int_{|\zeta - a| = \gamma} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}}$

To find the development of $f(z)$:

we consider the transformation $z = a + \frac{1}{z'}, \zeta = a + 1/\zeta'$

This transformation carries $|z-a| = r$ into $|\zeta'| = 1/\mathrm{r}$ with negative orientation.

This transformation can be applying the circle with negative orientation

$$f_2(z) = f_2(a + 1/z')$$
$$= \frac{-1}{2\pi i} \int_{|\zeta'|=1/\gamma} \frac{f(a+1/\zeta')(-1/(\zeta')^2) d\zeta'}{(1/\zeta' - 1/z')}$$
$$= \frac{1}{2\pi i} \int_{|\zeta'|=1/\gamma} \frac{f(a+1/\zeta') 1/\zeta'^2 d\zeta'}{(z' - \zeta')}$$
$$= \frac{-1}{2\pi i} \int_{|\zeta'|=1/\gamma} \frac{z' f(a+1/\zeta') d\zeta' (\zeta' - z')}{\zeta'}$$

Now we consider

$$\frac{1}{c' - z'} = \frac{1}{\zeta'(1 - z'/\zeta')}$$
$$= \frac{1}{\zeta'} \left(1 - \frac{z'}{\zeta'}\right)^{-1}$$
$$= \frac{1}{\zeta'} \left[1 + \frac{z'}{\zeta'} + \left(\frac{z'}{\zeta'}\right)^2 + \cdots\right]$$

$$\frac{1}{\zeta' - z'} = \frac{1}{\zeta'} + \frac{z'}{(\zeta')^2} + \frac{(z')^2}{(\zeta')^3} +$$

$$\therefore f_2(z) = -\frac{1}{2\pi i} \int_{|\zeta'|=1/\mathrm{r}} \frac{z'}{\zeta'} f(a+1/\zeta') \left[\frac{1}{\zeta'} + \frac{z'}{(\zeta')^2} + \frac{(z')^2}{(\zeta')^3} + \cdots ..\right] d\zeta$$
$$= \frac{-1}{2\pi i} \int_{|\zeta'|=1/\mathrm{r}} f(a+1/\zeta') \left[\frac{z'}{(\zeta')^2} + \frac{(z')^2}{(\zeta')^3} + \cdots \ldots \ldots\right] d\zeta$$
$$= \frac{-1}{2\pi i} \int_{|\zeta'|=1/\gamma} f(a+1/\zeta') \frac{z'}{\zeta'^2} d\zeta + \left(\frac{-1}{2\pi i}\right) \int_{|\zeta'|=1/\mathrm{r}} f(a+1/\zeta') \frac{z'^2}{(\zeta)^3} d\zeta + \cdots$$
$$= \sum_{n=1}^{\infty} B_n (z')^n$$

Where $B_n = \frac{-1}{2\pi i} \int_{|\zeta'|=1/\gamma} \frac{f(a+1/\zeta')}{(\zeta')^{n+1}} d\zeta'$

$\zeta = a + 1/\zeta' \Rightarrow \zeta - a = 1/\zeta' \Rightarrow \zeta' = \dfrac{1}{\zeta - a}$

$$d\zeta = \frac{-1}{(\zeta')^2} d\zeta' \Rightarrow d\zeta' = -(\zeta')^2 d\zeta$$

$$d\zeta' = \frac{-1}{(\zeta - a)^2} d\zeta$$

$$B_n = \frac{1}{2\pi i} \int_{|\zeta - a) = r} \frac{f(\zeta)(\zeta - a)^{n+1}}{(\zeta - a)^2} d\zeta$$

$$B_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{-n+1}} d\zeta$$

$$\therefore B_n = -A_n$$

$$\therefore f_2(z) = \sum_{n=1}^{\infty} A_{-n} \left(\frac{1}{z - a}\right)^n$$

$$= \sum_{n=+1}^{\infty} A_{-n} (z - a)^{-n}$$

$$= \sum_{n=+1}^{-\infty} A_n (z - a)^n$$

$$f(z) = \sum_{\infty}^{-\infty} A_n (z - a)^n$$

Exercises:

1.Prove that the Laurent development is unique.

Let $f(z) = \sum_{n=-\infty}^{\infty} P_n (z - z_0)^n$ is analytic in the annulus region, where $r_2 < |z - z_0| < r_1$ be the expansion of $f(z)$. Obtain in any mainor than the series is identical with Laurent series for $f(z)$.

Proof :

Let $f(z) = \sum_{n=-\infty}^{\infty} P_n (z - z_0)^n$ …………..(1)

be the $L - S$ of $f(z)$ about z_0 in $r_2 < |z - z_0| < r_1$

Then an $= \frac{1}{2\pi i} \int_c \frac{f(t)}{(t - z_0)^{n+1}} dt$

where $c: |t - z_0| = r,\ r_2 < r < r_1$

Given expansion of $f(z)$ is $f(z) = \sum_{m=-\infty}^{\infty} P_m (z - z_0)^m$ …………..(2)

To prove that $P_n = a_n$

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{(t-z_0)^{n+1}}$$

$$\therefore a_n = \frac{1}{2\pi i}\int_c \frac{1}{(t-z_0)^{n+1}} \sum_{m=\infty}^{\infty} P_m (t-z_0)^m dt$$

$$= \frac{1}{2\pi i}\sum_{m=\infty}^{\infty} P_m \int_c \frac{(t-z_0)^m}{(t-z_0)^{n+1}} dt \quad \ldots\ldots\ldots\ldots (3)$$

[Integrating term by term.

∴ the Series is uniformly convergent]

$C: |t - z_0| = \gamma$

$t - z_0 = re^{i\theta},\ 0 \leq \theta \leq 2\pi$

$dt = re^{i\theta} i d\theta$

Now sub in (3),

$$a_n = \frac{1}{2\pi i}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \frac{\left(re^{i\theta}\right)^m}{(re^{i\theta})^{n+1}} re^{i\theta} i d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \frac{\left(re^{i\theta}\right)^m}{(re^{i\theta})^{n}} d\theta$$

$$a_n = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} \left(re^{i\theta}\right)^{m-n} d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} P_m \int_0^{2\pi} r^{m-n} e^{i(m-n)\theta} d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} P_m r^{m-n} \int_0^{2\pi} e^{i(m-n)\theta} d\theta \quad \ldots\ldots\ldots\ldots (4)$$

case (i)

when $m \neq n$

$$\therefore \int_0^{2\pi} e^{i(m-n)B} d\theta = \left[\frac{e^{i(m-n)\theta}}{i^{(m-n)}}\right]_0^{2\pi}$$

$$= \frac{1}{i(m-n)}\left[e^{i(m-n)2\pi} - e^0\right]$$

$$= \frac{1}{i(m-n)}[1-1]$$

$$= 0 \quad \ldots\ldots\ldots\ldots\ldots (5)$$

case (ii)

When $m = n$

$$\int_0^{2\pi} e^{i(m-n)\theta} d\theta = \int_0^{2\pi} d\theta$$
$$= 2\pi$$

Sub (5) & (6) in (4),

$$a_n = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} P_m r^{m-n} (2\pi)$$
$$= \frac{1}{2\pi} [0 + 0 + \cdots + P_n r^0 (2\pi) + 0 + 0 \cdots]$$
$$= \frac{1}{2\pi} P_n 2\pi$$
$$a_n = P_n$$
$$[\because r^0 = 1]$$

Sub in (2) The given series (2) becomes Laurents series,

Hence the Laurents series of $f(z)$ is unique.

2. Show that the Laurent development of $(e^2 - 1)^{-1}$ at the origin is of the form

$$\frac{1}{2} - \frac{1}{2} + \sum_{1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

Where the numbers B_k are known as the Bernoulli numbers calculate B_1, B_2, B_3

Proof:

Consider the fun $f(z) = \frac{1}{e^z - 1}$

$$f(z) = \frac{1}{\left(1 + z + \frac{z^2}{2!} + \cdots\right) - 1}$$
$$= \frac{1}{z\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)}$$
$$\therefore f(z) = \frac{1}{z(g(z))}$$

The singularities of $f(z)$ is given by,

$$\therefore e^z - 1 = 0 \Rightarrow e^z = 1$$
$$z = 0, i2n\pi, n = 0, \pm 1, \pm 2 \ldots$$
$$= 0, i2\pi, i4\pi, \ldots$$

Consider the analus region is $0 < |z| < 2\pi$

Now consider the fun $\frac{1}{g(z)}$ which is defined an analytic in $0 < |z| < 2\pi$

$$\frac{1}{g(z)} = c_0 + c_1 z + c_2 z^2 +$$

$$\Rightarrow 1 = g(z)[c_0 + c_1 z + c_2 z^2 + \cdots]$$

$$1 = \left(1 + \frac{z}{2!} + \frac{z^3}{3!} + \cdots\right)(c_0 + c_1 z + c_2 z^2 + \cdots)$$

Comparing the co-efficient on z

$$\Rightarrow 0 = \frac{c_0}{2!} + c_1$$

$$\Rightarrow c_1 = \frac{-c_0}{2!}$$

$$c_1 = \frac{-1}{2!}$$

co-eff of $z^2 \Rightarrow 0 = \frac{c_1}{2!} + c_2 + \frac{c_0}{3!}$

$$0 = \frac{-1}{4} + c_2 + \frac{1}{6}$$

$$0 = c_2 - \frac{1}{12}$$

$$1/12 = c_2$$

co $-$ eff of $z^3 \Rightarrow 0 = c_3 + \frac{c_2}{2!} + \frac{c_1}{3!} + \frac{c_0}{4!}$

$$= c_3 + 1/24 - 1/12 + 1/24$$

$$0 = c_3$$

co-eff of $z^4 \Rightarrow 0 = c_4 + \frac{c_3}{2!} + \frac{c_2}{3!} + \frac{c_1}{4!} + \frac{c_0}{5!}$

$$= c_4 + 0 + \frac{1}{72} - \frac{1}{48} + \frac{1}{20}$$

$$0 = c_4 + 1/720$$

$$\Rightarrow c_4 = -1/720$$

Similarly,

co-eff of $z^5 \Rightarrow 0 = c_5 + \frac{c_4}{2!} + \frac{c_3}{3!} + \frac{c_2}{4!} + \frac{c_1}{5!} + \frac{c_0}{6!}$

$$0 = c_5$$

$$\therefore \frac{1}{g(z)} = 1 - \frac{1}{2}z + \frac{1}{12}z^2 - \frac{1}{720}z^4 + \cdots$$

$$f(z) = \frac{1}{zg(z)}$$

$$= \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots$$

$$f(z) = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{B_k}{2^{k!}}z^{2k-1}$$

Comparing L.H.S = R.H.S

$$k = 1, \frac{B_1}{2!} = \frac{1}{12} \Rightarrow B_1 = \frac{1}{6}$$

$$k = 2, \frac{B_2}{4!} = \frac{1}{720} \Rightarrow B_2 = 1/30$$

$$k = 3, \frac{B_3}{6!} = 0 \Rightarrow B_3 = 0$$

3.Expand the Laurents Series:

Then function is $\frac{1}{(z-1)(z-2)}$ is valid in

(i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 2$ (iv) $0 < |z - 1| < 1$

Proof:

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

Case (i) :- $|z| < 1$

$$f(z) = \frac{1}{-2(1 - z/2)} - \frac{1}{-(1 - z)}$$

$$= \frac{1}{1 - z} - \frac{1}{2(T - z/2)}$$

$$f(z) = (1 - z)^{-1} - \frac{1}{2}(1 - z/2)^{-1}$$

$$= (1 + z + z^2 + \cdots) - \frac{1}{2}(1 + z/2 + (z/2)^2 + \cdots)$$

$$f(z) = 1/2 + z(1 - 1/2^2) + z^2(1 - 1/2^3) + \cdots$$

The series is valid when $|z/2| < 1, |z| < 1$

$|z| < 2, |z| < 1$

Case(ii): $1 < |z| < 2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$
$$= \frac{1}{-2(1-z/2)} - \frac{1}{z(1-1/z)}$$
$$= \frac{-1}{2}(1-z/2)^{-1} - \frac{1}{z}(1-1/z)^{-1}$$
$$= -1/2(1 + z/2 + (z/2)^2 + (z/2)^3 + \cdots)$$
$$-1/z(1 + 1/z + (1/z)^2 + \cdots)$$

This expansion is valid

when $|z/2| < 1, |1/z| < 1$

$|z| < 2, |z| > 1$

$\therefore 1 < |z| < 2$

Case(iii): $|z| > 2$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$
$$= \frac{1}{z(1-2/z)} - \frac{1}{z(1-1/z)}$$
$$= \frac{1}{z}(1-2/z)^{-1} - 1/z(1-1/z)^{-1}$$
$$= \frac{1}{z}(1 + 2/z + (2/z)^2 + (2/z)^3 + \cdots)$$
$$\therefore f(z) = \frac{1}{z^2}(z-1) + \frac{1}{z^3}(1 + 1/z + (1/z)^2 + \cdots) + \frac{1}{z^4}(8-1) + \cdots$$

This expansion is valid

when $|2/z| < 1 \Rightarrow 2 < |z|$

$\therefore |z| > 2$

Case(iv):: $0 < |z-1| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

Let $t = z - r \ \Rightarrow t + 1 = 2,\ 0 < |t| < 1.$

$$\therefore f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{t(t-1)}$$
$$= \frac{1}{-t(1-t)}$$
$$= -1/t(1+t+t^2+\cdots)$$
$$= \frac{-1}{z-1}(1+(z-1)+(z-1)^2+\cdots)$$

This expansion is valid. when $|t| < 1, t \neq 0$

$\therefore z-1 \neq 0$ and $0 < |z-1| < 1$

4.What is the co-eff of z^7 in the Taylor's development of $\tan Z$.

Solution:

$$f(z) = \tan z$$
$$\tan z = c_1 z + c_2 z^3 + c_3 z^5 + \cdots$$
$$\frac{\sin z}{\cos z} = c_1 z + c_2 z^3 + c_3 z^5 + \cdots$$
$$\frac{z - \frac{z^3}{3!} + \frac{2^5}{5!} + \cdots}{1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots} = c_1 z + c_2 z^3 + c_3 z^5 + \cdots$$

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)(c_1 z + c_2 z^3 + c_3 z^5 + \cdots)$$

co-eff of $z \Rightarrow c_1 = 1$

$\text{co} - \text{eff of } z^2 \Rightarrow 0$

$\text{co} - \text{eff of } z^3 \Rightarrow -1/6 = c_2 + \frac{c_1}{2!}$

$c_2 = -2/3$

$\text{co} - \text{eff of } z^5 \Rightarrow 1/120 = c_3 + \frac{c_2}{21}$

$c_3 = \frac{21}{120}$

$\text{co} - \text{eff of } z^7 \Rightarrow 1/7! = c_4 = \frac{c_2}{4!}$

$$\frac{1}{1540} = c_4 - \frac{2}{72}$$
$$c_4 = \frac{141}{5040}$$

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